UNM Dept. of Mathematics and Statistics
MS/PhD Qualifying Examination Fall 2024
Ordinary and Partial Differential Equations

Of the following six problems, please do five of your choice. Please start each problem on a new page labeled with the problem number and your secret ID. Do not attempt to submit solutions of all six problems; if you do so, only the first five will be graded.

(1) (20 points total) Locate all equilibrium points of the following ODE system

$$\begin{cases} x' &= x(1-x^2-y^2) \\ y' &= y(1-x^2-y^2) \end{cases}$$

Sketch the phase plane diagram, and discuss the stability properties of equilibrium points.

(2) (20 points total) Find the appropriate Lyapunov function to determine the type of the nonlinear stability of the equilibrium point at the origin of the following ODE system

$$\begin{cases} x' &= -x+2y+2yx, \\ y' &= \frac{x}{2}-y-\frac{x^2}{2}-8y^3. \end{cases}$$

Determine the domain of the stability in (x, y) plane.
Hint: you can try to use a quadratic function of x and y as the candidate for Lyapunov function.

(3) (20 points total) Consider the nonlinear ODE system

$$\begin{cases} x' &= y+2x-x^3-\frac{3}{2}xy^2, \\ y' &= 2y-x-x^2y-\frac{3}{2}y^3. \end{cases}$$

(a) (10 points) Show that the nonlinear system has at least one periodic orbit in the annulus $\sqrt{4/3} \le r \le \sqrt{2}$ (for now use the fact, that the only critical point is the origin).
(b) (5 points) Show that there is exactly one periodic orbit inside the annulus.
(c) (5 points) Prove that there are no critical points but the origin.
Hint: use polar coordinates.

(4) (20 points total)
a) (4 points) For each of the following PDEs, state whether and why the PDE is linear, semilinear, quasilinear, or fully nonlinear:
i) $u_{tt}+u_t-u=x^2$
ii) $u_t+uu_x+u_{xxx}=0$
b) (8 points) Precisely state the following properties of harmonic functions:
- mean-value property
- maximum principle

c) (8 points) Consider the following function

$$u(x_1, x_2) = \ln(x_1^2 + x_2^2), \qquad (x_1, x_2) \in \bar{U} \subset \mathbb{R}^2,$$

where $\bar{U}$ is the closure of the annulus $B(0,1) - B(0, 0.5)$, i.e. the region between two closed balls centered at 0 and with radii 0.5 and 1.

- Verify that u is harmonic.
- Verify that both maximum principle and minimum principle hold for u. Justify your answer.

(5) (20 points total) Solve the following problem using the method of characteristics:

$$\begin{aligned} u_{x_1} u_{x_2} - u &= 1, \qquad && x_1 \in \mathbb{R}, \quad x_2 \geq 0, \\ u &= x_1, && x_1 \in \mathbb{R}, \quad x_2 = 0. \end{aligned}$$

(6) (20 points total) Use the methods of separation of variables and eigenfunction expansion and solve the following inhomogeneous initial-boundary value problem for the heat equation

$$u_t - 3u_{xx} = 0, \quad x \in (0,3), \quad t > 0,$$
$$u(x,0) = \sin(\frac{7\pi x}{3}),$$
$$u(0,t) = u(3,t) = 0.$$