In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2024

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained. Do not use calculators, cell phones, or other electronic devices.*

Problem 1. (20 pts) Suppose you have a deck of cards numbered from 1 to N. After shuffling the deck, you start picking cards one by one from the top. The first card is always kept. For each subsequent card, you keep it if it is greater than the current card in hand; otherwise, you discard it.

(a) What is the expected number of cards you will keep?

(b) what is the variance of the number of cards you will keep?

Problem 2. (20 pts) In a raffle with 900 tickets, 9 people buy 100 tickets each. If there are 6 winning tickets drawn at random, find the probability that

(a) one person gets all 6 winning tickets.

(b) there are 6 different winners.

Problem 3. (10 pts) X and Y are independent exponential random variables with mean 1. If $V = X + Y$, find the mean and variance of X given $V = 10$.

Problem 4. (30 pts) Assume that we have observed $x_1, x_2, \ldots, x_n | p \overset{iid}{\sim} Binomial(100, p)$ and that we are interested in estimating $\tau(p) = (1-p)^{100}$. Recall that the probability mass function (pmf) of a $Binomial(100, p)$ is

$$f(x|100,p) = \begin{cases} \binom{100}{x} p^x (1-p)^{100-x} & \text{if } x = 0, 1, \ldots, 100 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the Maximum Likelihood Estimator of $\tau(p)$.

(b) Verify that

$$\hat{\tau}_{naive} = I(x_1 = 0) = \begin{cases} 1 & \text{if } x_1 = 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a unbiased estimator of $\tau(p)$.

(c) Show that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for p.

(d) Show that $T|X_1 = 0 \sim Binomial(100(n-1), p)$

(e) Find the Uniform Minimum Variance Unbiased Estimator (UMVUE) of $\tau(p)$. *Hint: Rao-Blackwellize the estimator in (b).*

Problem 5. (20 pts)

Suppose $X_1, \ldots, X_n$ are a random sample from the $Gamma(\alpha, \beta)$ for unknown $\alpha > 0$ and $\beta > 0$. Recall that the probability density function (*pdf*) of a $Gamma(\alpha, \beta)$ is $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$.

(a) Find the distribution of $\sum_{i=1}^{n} X_i$.

(b) Suppose $\alpha = 1$, find a pivotal quantity for β and determine its distribution.

(c) Find a_1 and a_2 such that

$$\sqrt{n}\left(\frac{n}{\sum_{i=1}^{n} X_i} - a_1\right) \xrightarrow{d} N(0, a_2).$$