# Quasi-linear differential algebraic equations : an algebraic study

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We present some investigations about quasi-linear differential algebraic equations (DAEs) :

$$A(t, y(t))dy/dt = b(t, y(t))$$

$y = (y_1, \ldots, y_n)^T$ is the unknown function . $A$ and $b$ are respectively a $m \times n$ matrix and a $n$-vector the coefficients of which are polynomials $\in K[t, y]$ ; $K$ is a field containing the rational numbers. We focus our study on the case where rank$(A(t, y))$ is lower than $n$ for all $(t, y)$, which implies the $y_i$'s are constrained to be on an algebraic variety. Typical such systems are the *semi-explicit* DAEs:

$$\left\{ \begin{array}{rcl} dy_1/dt & = & f(t, y_1, y_2) \\ 0 & = & g(t, y_1, y_2) \end{array} \right.$$

In the square case ($m = n$), referring to Gear, we define the *differential index* as the minimal number of differentiations needed to get a full implicit ODE, that is a relation $C(t, y)dy/dt = d(t, y)$ with rank($C$)$= n$ for most $(t, y)$. Then we give a geometrical interpretation of this index to obtain global results, related to the formal structure of the system only .

Computing a relations matrix $R$ of size $(n - r) \times r$ ($r = \text{rank}A$) among the rows of $A$, satisfying $RA = 0$, we obtain relations among the components of $b$ : these are the algebraic constraints $y$ has to stand. Let $\mathcal{G}_0$ be the variety related to the constraints.

Then we notice that if $y(t)$ is a function on the corresponding variety, $dy/dt$ must belong to the tangent space to this variety. So we differentiate the algebraic relations : it leads to quasi-linear equations, which we put in the initial system. For example let $g(t, y)$ be a polynomial that vanishes on $\mathcal{G}_0$ ; we add the folowing equation to the quasi-linear system :

$$\sum_{i\leq n} \frac{\partial g}{\partial y_i} dy_i/dt = -\frac{\partial g}{\partial t}$$

We do the same with this new quasi-linear DAE. If the rank of the matrix is $< n$, new constraints appear and we have a subvariety of $\mathcal{G}_0$.

We show that this iterative process always stops after a finite number of steps that is $\leq n$ : this number is equal to the differential index.

So we got a decreasing sequence of algebraic varieties, the limit of which is the variety $\mathcal{G}$ : the graphs of solutions lie on it. We define regular points on $\mathcal{G}$ as those who maximize the value of rank($C$). Around such points the system can be put in upper triangular form with a diagonal made of non-zero polynomials. A way to perform that is to use a fraction-free gaussian elimination. It enables us to answer the questions of existence and uniqueness of solution(s) passing through regular points.

We must notice that the index is not intrinsic since it depends on the different irreducible components of $\mathcal{G}$. The complex problem of decomposing an ideal into its primary components has to be solved to obtain an exhaustive algorithm. We are programming the irreducible case (all the varieties of the sequence are irreducible) in MapleV.

The future and actual part of the work is the location and definition of singular points. This is not exactly similar to what happens with implicit quasi-linear ODE (when rank($A$)=n).

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