Algebraic approach for quasi-linear constrained differential equations

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Abstract

To study the existence of solutions to constrained quasi-linear differential equations with polynomial coefficients, we use commutative algebra. First we emphasize that the possible solutions have to stand algebraic constraints, and we give a way to compute those constraints. It means that the study has to be carried out on an algebraic subvariety of the ambient space. We define the so-called index of the differential-algebraic equation, by inspiring from Reich or Rheinboldt work. It is interpreted as the minimal number of differentiations needed to obtain an underlying ODE of maximal rank. The geometrical definition of the index is extended to over or subdetermined systems.

Then we define regular points and eventually answer the questions of existence and uniqueness of solutions at such points, using a reduced and simpler local form of the system. An effective algorithm summarizes our approach.

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1 Introduction

1.1 The Differential Index

Generally speaking, Differential Algebraic Equations (DAEs) are a mixture of algebraic and differential equations, which can be written :

$$F(t, y, dy/dt) = 0.$$

y is the unknown vector-function with $n \geq 1$ components, F is a function of the 3 variables (t, y, p), p having n components . We suppose that $\partial F/\partial p$ has rank $\leq r$ for all (t, y, p) *and in particular do not assume* $r = n$. In the case $r < n$, some algebraic equations, relating only t and y, are hidden in the DAE.

Those systems were first studied in a numerical context (see Gear [5] [6]) : the definition of a *differential index*, characterizing the difficulty of integrating numerically such equations, was the first step to understand the difference between DAEs and implicit non-linear DEs. Gear defined the index as being *the minimal number of differentiations needed to obtain a system invertible with respect to the derivative* y'. This did not provide existence, nor a bound for the value of the index.

Nowadays this index is related only to the *structure* of the system. Some authors, including Reich [12], Szatkowski [13][14], Rheinbolt and Rabier [11], have given an intrinsic definition of it, using differential geometry. This work was replaced in the formal theory of DEs by Levey [15] and Piirilä & Tuomela [8], following Pommaret's indications [9]. Their definition is more accurate since it makes the understanding clearer : the differential index is the number of prolongations-projections to apply to the system in order to get an equivalent one that is *formally integrable* (see a definition in [15] or [4]).

Indeed the formal way of treating differential-algebraic systems is known since Cartan or Riquier, in the beginning of the century. Moreover, a theorem of Goldschmidt ensures the existence for sufficiently smooth functions F, and gives a bound related to the dimension of the jet-space used and the dimension of the manifold implicitely defined by $F(t, y, p) = 0$ in this space.

But none of those approaches leads to an efficient and effective method to compute the index and put the system into integrable form. This may be due to their wide generality (Goldschmidt's results are available for PDEs too).

The simplest case is the linear one, which often appears in practical applications :

$$A(t)y' + B(t)y = f(t).$$

Effective results have been obtained by Campbell [3]. Kunkel and Mehrmann propose an algorithm in [7], revisited by Quere in [10]. A canonical form inspired from the Kronecker form of pencils of matrices is defined for linear differential-algebraic equations. Our main purpose will be how to generalize those results to **quasi-linear systems** :

$$A(t,y)dy/dt = b(t,y) \tag{1.1}$$

For every (t, y), rank$A(t, y)$ is lower than a fixed value $r \leq n$. There will be algebraic constraints on y, i.e. equations of the type $g(t, y) = 0$ where g is a polynomial, if (and only if) $r < n$.

In the sequel, we shall detail a geometrical approach for those systems. We shall see the differential index can be computed simply, using linear and commutative algebra only ; in particular no differential algebra is needed. Then we shall generalize our results to overdetermined or subdetermined systems. The proof of integrability will be done directly, without making use of general results of Goldschmidt. Since the constraints equations encountered will be algebraic, we study DAEs on the base field of complex numbers, which is not usually done in the existent litterature. Working with an algebraically closed field is natural when dealing with algebraic equations and it simplifies the tests for emptyness of the varieties.

1.2 Problem

The main problem we are concerned with is to answer the question :

Let $(t_0, y_0) \in \mathbb{C}^{n+1}$. Does (1.1) have a ***solution*** *at this point, i.e. is there a differentiable vector function $y(t)$ defined in a neighborhood V of t_0, such that (1.1) holds for all $t \in V$ and $y(t_0) = y_0$?*

At first we show how to compute the algebraic constraints (t_0, y_0) will have to stand. We will then inject their derivatives to the differential system, and compute again the rank of the matrix and the new constraints. It will lead us to an iterative algorithm, which stops after a number of iterations equal to the index of the DAE. At this time we will know the form of the system at regular points (see their definition in part 4) and in the end obtain a theorem of existence and uniqueness of solutions near such points. The whole study will be interpreted in a geometrical context.

2 Global study

2.1 Preliminaries

In the following we shall use capitals for indeterminates and small letters when we specialize indeterminates into particular values. Y is the formal indeterminate $Y = (Y_1, \ldots, Y_n)$, $n \in \mathbb{N}^*$, T is the indeterminate for time. The symbol ^{T} following an expression will denote its *transpose*.

In (1.1), we shall suppose that A and b are respectively a square $n \times n$-matrix and a n-column vector, both having coefficients in the ring of multivariate polynomials $\mathbb{C}[T, Y]$.

Definition 2.1. *Let M be a matrix with coefficients in $\mathbb{C}[T, Y]$ and $\mathcal{G}$ an algebraic variety of $\mathbb{C}^{n+1}$. The **rank** of M is the usual value of its rank when considered in the field of multivariate fractions on $\mathcal{G}$.*

To distinguish it from the rank, we call the *local rank* of M at $(t^*, y^*) \in \mathcal{G}$ the value of the rank of the complex matrix $M(t^*, y^*)$. It is straightforward that for most points of $\mathcal{G}$, the local rank is equal to the rank of M, and that the rank defined above is the maximal value of local ranks on $\mathcal{G}$.

For example, the rank of

$$M = \begin{pmatrix} Y_1 - 1 & T - Y_2 \\ Y_2 - 3Y_1 & 1 \end{pmatrix}$$

equals 2 in $\mathbb{C}^3$ and 1 in the variety

$$\{(T, Y_1, Y_2) \in \mathbb{C}^3 \ / \ \det M(T, Y_1, Y_2) = 0\}.$$

2.2 Algebraic constraints

Suppose there is a solution to (1.1) at a given point $(t^*, y^*) \in \mathbb{C}^{n+1}$, i.e. a differentiable function $y(t)$ defined around t^* such that $y(t^*) = y^*$ and (1.1) is true. Then the scalar linear system $A(t^*, y^*)p = b(t^*, y^*)$ possesses at least one solution $p \in \mathbb{C}^n$. We are led to the study of existence of solutions to the following linear equation, parametrized by the indeterminates Y and T :

$$A(T, Y)p = b(T, Y) \tag{2.2}$$

Let $\mathcal{L}$ be the $\mathbb{C}[T, Y]$-module generated by the *rows* of $A(T, Y)$. We build an exact sequence of $\mathbb{C}[T, Y]$-modules :

$$(0) \longrightarrow K \longrightarrow \mathbb{C}[T, Y]^n \longrightarrow \mathcal{L} \longrightarrow (0)$$

If $(e_1, \ldots, e_n)$ is a basis of $\mathbb{C}[T, Y]^n$, the homomorphism $\mathbb{C}[T, Y]^n \longrightarrow \mathcal{L}$ is given by $e_j \mapsto L_j \in \mathbb{C}[T, Y]^n$, where L_j is the j-th row of A. Its kernel is the $\mathbb{C}[T, Y]$-module K.

Let r be the rank of A. If $r < n$, the rows of A are not $\mathbb{C}[T, Y]$-linearly independent. Let R be a *relations matrix* for the rows of A (see Brown [2] p116), the rows of which form a basis of K : it implies that R is a polynomial $(n - r) \times n$-matrix.

In the following example,

$$A(T,Y) = \begin{pmatrix} Y_1+1 & Y_2-Y_1^2 \\ T(Y_1+1) & T(Y_2-Y_1^2) \end{pmatrix}$$

we can choose $R = (-T \;\; 1)$ so that

$$K = \{-u(T,Y)T.e_1 + u(T,Y).e_2 \;/\; u \in \mathbb{C}[T,Y]\}.$$

We obtain then the lemma

Lemma 2.1. *Let R a relations matrix for the rows of A : if system (2.2) has a solution at $(t^*,y^*) \in \mathbb{C}^{n+1}$ then*

$$R(t^*,y^*)b(t^*,y^*) = 0.$$

This follows directly from the formal relation $RA = 0$.

Hence the solutions to (1.1) are constrained to be on the algebraic sub-variety of $\mathbb{C}^{n+1}$ given by the vanishing of the $n-r$ polynomial coordinates of the vector $R(T,Y)b(T,Y)$. In the next lemma we establish that this variety depends only on A and b :

Lemma 2.2. *Let A be a square $n \times n$-matrix, with coefficients in $\mathbb{C}[T,Y]$ and b a vector of $\mathbb{C}[T,Y]^n$. Let R be a relations matrix among the rows of A. The algebraic variety defined by $\mathcal{G}_0 = \{(t,y) \in \mathbb{C}^{n+1} \;/\; R(t,y).b(t,y) = 0\}$ would be the same for another choice of relations matrix.*

Proof: we have seen that such a matrix was the transpose of a basis of $K = \ker A^T$, a sub-module of the free $\mathbb{C}[T,Y]$-module $\mathbb{C}[T,Y]^n$. K depends only on A. If R' is another relations matrix of A, there is a linear link between R and R' :

$$R' = \Lambda R$$

where Λ is a square matrix with determinant invertible in $\mathbb{C}[T,Y]$. We conclude that $R'b(t,y) = 0$ if, and only if, $Rb(t,y) = 0$. We have then proved the lemma.

It can be extended when the ring of coefficients is a quotient $\mathbb{C}[T,Y]/\mathcal{I}$ and $\mathcal{I}$ is a prime ideal in $\mathbb{C}[T,Y]$, because $(\mathbb{C}[T,Y]/\mathcal{I})^n$ is a free $\mathbb{C}[T,Y]/\mathcal{I}$-module in that case. $\square$

Now we come back to the study of (2.2).
Let $\mathcal{G}_0 = \{t,y) \in \mathbb{C}^{n+1} \;/\; R(t,y).b(t,y) = 0\}$: if $y(t)$ is a differentiable function such that

$$\forall t \;\; (t,y(t)) \in \mathcal{G}_0,$$

we have

$$\frac{dy}{dt}(t,y(t)) \in T_{(t,y(t))}\mathcal{G}_0.$$

Obviously we suppose that this last expression has a meaning, which is fortunately the case for most points in an algebraic variety. We must add a system of equations describing the tangent space to $\mathcal{G}_0$ to system (2.2). If $g(T,Y)$ is a non-zero polynomial that vanishes on $\mathcal{G}_0$, we put the new equation

$$\sum_j \frac{\partial g}{\partial Y_j} p_j = -\frac{\partial g}{\partial T}.$$

It is linear with respect to the p_j's and the coefficients belong to $\mathbb{C}[T,Y]$.

So we got an augmented linear system

$$A_1(T,Y)p = b_1(T,Y),$$

for which we study once again existence of solutions : this time we have to consider polynomials in the quotient ring $\mathbb{C}[T,Y]/\mathcal{I}_0$, where $\mathcal{I}_0$ is the radical ideal related to $\mathcal{G}_0$. In order to preserve the property that the local rank is equal to the rank for most points in $\mathcal{G}_0$, this quotient ring must keep an integral domain. It is false when $\mathcal{I}_0$ is not prime : this means $\mathcal{G}_0$ is made of a finite number of irreducible components, so that we have to run out separate computations on each of them.

In that section we have explained an iterative process that computes the constraints in an intrinsic way. It will allow us to answer the following questions :

- Does system (1.1) have any solution ?
- On which irreducible sub-variety(ies) are the solutions to (1.1)?
- At a point through which a solution passes, is this solution unique ?

3 Index of a quasi-linear DAE

3.1 Definition

In this section we suppose that all the algebraic varieties we consider are irreducible. We shall see how to deal with the general case in the last section (cf algorithm 5.1).

For a system like (2.2), we define the following sequences, generated in the way described in the previous section :

- Algebraic variety which we study the system on : $\mathcal{G}_i$, of codimension c_i. $\mathcal{G}_{i+1}$ is obtained by adding the equations of $T\mathcal{G}_i$ to the system $A_i p = b_i$ and computing all of its hidden algebraic constraints.
- Prime Ideal related to $\mathcal{G}_i$: $\mathcal{I}_i$.
- Matrix of the differential part : $A_i(T,Y)$. We put $A_0 = A$. We define r_i as the rank of A_i, computed on $\mathcal{G}_i$: that is the maximum value of the local rank on this variety.

- Second member : $b_i(T, Y)$, with $b_0 = b$.
- Number of free components of p : u_i. It is the number of coordinates which have no equation to satisfy.

We explain how to handle these sequences on the following example (the DAE is noted $(A|b)$) :

$$\left(\begin{array}{cccc|c} Y_1+1 & Y_2-Y_1^2 & 1+Y_3 & & -TY_1+2Y_1Y_2^3 \\ T(Y_1+1) & T(Y_2-Y_1^2) & T(1+Y_3) & & -Y_3 \\ 2T(Y_1+1) & 2T(Y_2-Y_1^2) & 2T(1+Y_3) & & -2Y_3 \end{array}\right)$$

A relations matrix is

$$R = \begin{pmatrix} T & -1 & 0 \\ 2T & 0 & -1 \end{pmatrix}$$

Here we have $r_0 = 1$ so $c_0 + u_0 = 2$. $\mathcal{I}_0 = (T^2Y_1 - 2TY_1Y_2^3 - Y_3)\mathbb{C}[T, Y]$, whence $c_0 = 1 = u_0$. The system can be reduced to a simpler form on $\mathcal{G}_0 = \{(T, Y) \ / \ T^2Y_1 - 2TY_1Y_2^3 - Y_3 = 0\}$, by taking (Y_1, p_1) as formal parameter, choosing the first row in the initial system and adding the derivative of the constraint.

$$\left\{\begin{array}{rcrcl} (Y_2-Y_1^2)p_2 & + & (1+Y_3)p_3 & = & -TY_1+2Y_1Y_2^3-(1+Y_1)p_1 \\ -6TY_1Y_2^2p_2 & - & p_3 & = & 2Y_1(Y_2^3-T)+(2Y_2^3-T)Tp_1 \end{array}\right.$$

The second equation forces derivatives (p_1, p_2, p_3) to stay on the tangent space to $\mathcal{G}_0$. The rank of the differential system satisfied by (p_2, p_3) is 2, so that there is no relation among its rows. Solutions have no additional constraint to stand. The parameters for $i = 1$ are $r_1 = 2,\ u_1 = 1,\ c_1 = 1$.

From their definitions, we can notice that (r_i) and (u_i) are increasing sequences : A_i contains all the lines of A_{i-1}, and u_i is the number of redundant equations in the system. Intuitively we can care about existence of solutions when we have one information for each derivative, which leads to the following definition :

Definition 3.1 (Index of a quasi-linear DAE). *Let (A, b) be a quasi-linear DAE with polynomial coefficients. Let $(\mathcal{G}_i)$ the associated sequence of irreducible algebraic varieties. If $r_0 + u_0 < n$, we call* ***index*** *of (A, b) the smallest $i \in \mathbb{N}$ such that $\mathcal{G}_i = \mathcal{G}_{i-1}$. Otherwise we say index is zero.*

In the previous example, $\mathcal{G}_1 = \mathcal{G}_0$ since after one iteration no new constraint was found. The index of the system is 1.

3.2 A characterization of the index

We are going to prove that this index is always defined and lower than n. Then we shall see it is a generalization of the *differential index* introduced by Gear in [5].

Theorem 3.1. *Let (A, b) be a quasi-linear DAE with polynomial coefficients. There exists a $k \in \{0, \dots, n\}$ such that the sequence $(\mathcal{G}_i)$ is stationnary for $i \geq k-1$ and strictly decreasing for $i < k-1$. Further, k is characterized as being the smallest i satisfying*

$$u_i + r_i = n.$$

Proof: $(\mathcal{G}_i)$ is a decreasing sequence of algebraic varieties defined with *noetherian rings*, so we can assert it becomes stationnary. To show that $k \leq n$, we have to prove that as soon as we find a j such that $\mathcal{G}_{j+1} = \mathcal{G}_j$, the sequence does not decrease any more.

Suppose $j \in \mathbb{N}$ and $\mathcal{G}_{j+1} = \mathcal{G}_j$. It means the relations matrices of the rows of A_j are whether null matrices, and do not induce algebraic constraints on $b_j(T, Y)$, or induce algebraic constraints whose derivatives are linearly dependent on the rows of A_j, which causes $u_{j+1} > u_j$ but does not change $\mathcal{G}_j$. Hence the supplementary equations in the system $A_{j+1}p = b_{j+1}$ are consequences of the system $A_j p = b_j$ so that they have the same solutions. The differential system on $\mathcal{G}_{j+1}$ is equivalent to this on $\mathcal{G}_j$, so

$$\mathcal{G}_{j+2} = \mathcal{G}_{j+1}.$$

We have proved consequently that index could not be greater than n.

Now let us show the second assertion. Suppose $k \neq 0$ and let $j \in \{1, \dots, n\}$: if

$$u_{j-1} + r_j < n$$

the system on $\mathcal{G}_j$ provides $n - r_j - u_{j-1}$ new algebraic constraints for (T, Y). If all of them belong to $\mathcal{I}_{j-1}$, $u_j = u_{j-1} + n - r_j - u_{j-1}$ whence

$$r_j + u_j = n.$$

As no new equation has been found, we have $\mathcal{G}_j = \mathcal{G}_{j-1}$ and j is the index. If some of these algebraic constraints do not belong to $\mathcal{I}_{j-1}$, $u_j < n - r_j$, $\mathcal{G}_j \neq \mathcal{G}_{j-1}$ and j is not the index. □

3.3 Connection with the differential index

Now we recall the definition of the *differential index* (Gear [5]).

Definition 3.2. *Consider an implicit differential equation*

$$F(t, y, y') = 0$$

where F is a sufficiently differentiable map from $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$, and $y' = dy/dt$. The ***differential index*** *is defined by :*

1. *If $\partial F/\partial y'$ is invertible, index is* 0.

2. *Otherwise the index is the smallest $s \in \mathbb{N}$ such that the differential equations $dF/dt = 0$, $d^2F/dt^2 = 0, \ldots, d^sF/dt^s = 0$ give an implicit expression of all the components of y' as functions of y and t only.*

Let's notice this definition does not ensure index always exists, nor underlines the problem of possible free (or arbitrary) coordinates. We shall see that the index we have introduced is equal to Gear's one when $r_k = n$: it is the maximal rank of every first order differential system that one may obtain starting from the initial one (1.1) and using differentiation and usual operations in the module of functions.

Indeed at the j-th iteration A_j is simply the jacobian of $A_j.p - b_j$ with respect to p, the formal representation of y'. If we suppose $r_k = n$, $u_k = n - r_k = 0$ so that $u_j = 0$ for all $j \in \{0, \ldots, n-1\}$ (u_j is an increasing sequence). For $j < k$, $r_j + u_j = r_j < n$ since j is not the index. k is really the smallest s such that p is a function of y and t : that is to say k is the differential index.

If $u_k > 0$, $r_k = n - u_k < n$ but one cannot find new algebraic or differential equation from the system $A_k(T,Y)p = b_k(T,Y)$ and the equations describing $\mathcal{G}_k$. The converse would contradict the assertion that k is the index of (1.1). (r_j) is stationnary for $j \geq k$ so there is no quasi-linear differential equation of rank $> r_k$ in this system.

This can be reformulated in the formal theory : one would say the system is *involutive* or *formally integrable* since no new differential or algebraic relation can be performed with the equations obtained at the k-th iteration. In the context of ODEs or DAEs, the formal jet-theory says nothing more than the direct study we have used. That is the reason why we prefer a simpler geometrical language. The formal theory is of greater interest when dealing with PDEs (see [8]).

4 Form of the system at regular points

We still assume the $\mathcal{G}_i$'s are irreducible.

4.1 Local form at regular points

We first define *regular points* of a quasi-linear DAE. We compute an upper triangular form available in a neighborhood of such points. The questions of existence and uniqueness will be easier to answer then.

Definition 4.1. *Let (A, b) an index-k quasi-linear polynomial DAE. A point $(t_0, y_0) \in \mathbb{C}^{n+1}$ is said to be regular if $(t_0, y_0) \in \mathcal{G}_k$ and at this point local rank of A and rankA coincide :*

$$rank(A_k(t_0, y_0)) = r_k = rank(A_k).$$

At regular points one can explicit r_k components of p. If $u_k > 0$ the u_k other components of p are arbitrary differentiable functions of T.

We examine a 3 dimensional example before proving the previous statement :

$$\begin{pmatrix} T & 2Y_2 & 0 \\ -3 & 0 & 2Y_3 \\ (T-3) & 2Y_2 & 2Y_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 2Y_1 + Y_2 \\ 0 \\ 5Y_1 + Y_2 - Y_3^2 \end{pmatrix}$$

We deduce from this system that

$$\mathcal{G}_0 = \{Y_3^2 - 3Y_1 = 0\}.$$

The equation of its tangent space is in the initial system, which implies $u_1 = 1$ so $\mathcal{G}_1 = \mathcal{G}_0$: we have an index-1 DAE. The regular points satisfy rank$A_1 = 2$ and for example we can take $y_1(t) = \varphi_1(t)$ as parameter. The system is reduced to the form :

$$\begin{cases} 2y_2p_2 & = & y_2 - t\varphi_1'(t) + 2\varphi_1(t) \\ \qquad 2y_3p_3 & = & 3\varphi_1(t) \end{cases}$$

The matrix is now diagonal and the solutions to this last system whose initial conditions belong to $\mathcal{G}_1$ are the *regular* solutions of the system $A(t,y)y' = b(t,y)$. The other are called *singular* solutions : for them the jacobian matrix with respect to y' is a singular matrix at each time. Their existence is discussed in paragraph 4.3.

We summarize the results about reduction to a simpler form in the following theorem :

Theorem 4.1. *Let (A,b) be an index-k quasi-linear polynomial DAE and $(t,y) \in \mathbb{C}^{n+1}$ a regular point. Suppose A_k is a $l \times n$-matrix ; then there exists a matrix M of size $n \times l$, with polynomial coefficients, such that a dynamical system of maximal rank underlying (1.1) is*

$$(MA_k)p = \begin{pmatrix} A_d & A_u \\ 0 & 0 \end{pmatrix} p = Mb_k = \begin{pmatrix} b_d \\ 0 \end{pmatrix}.$$

$A_d \in Gl\,(r_k, \mathbb{C}(T,Y)/\mathcal{I}_k)$ has polynomial coefficients and is upper triangular. rank$A_k = r_k$ in a neighborhood of (t,y). A_u is a $r_k \times u_k$-matrix with polynomial coefficients. b_d is the second member of the differential part, it is a vector of size r_k with polynomial coefficients.

When $u_k = 0$, A_u vanishes, and so do the u_k equations written $0 = 0$.

Proof: in the final system

$$A_k(T,Y)p = b_k(T,Y)$$

the coefficients of A_k and b_k are polynomials defined modulo $\mathcal{I}_k$.
rank$A_k(t, y) = r_k \leq n$, so that there exists a $r_k \times r_k$ minor of A_k that does not vanish in a neighborhood of (t, y). Applying a fraction-free linear elimination, we obtain exactly the desired form : the transformation matrix M has elements in the quotient ring of polynomials modulo $\mathcal{I}_k$. If some of the $n - r_k$ last components of Mb_k did not belong to $\mathcal{I}_k$, that is they would not be reduced to zero modulo this ideal, there would be a contradiction because no new relation can be deduced from the system at iteration k.$\square$

4.2 About local existence and uniqueness

There will be no solution if no point verifies all the algebraic constraints :

Lemma 4.1. *Let (A, b) be a quasi-linear DAE and $\mathcal{T}_j = \mathcal{I}_j \cap \mathbb{C}[T]$. If for some j, $\mathcal{I}_j = (1)$ or $\mathcal{T}_j \neq (0)$ then the system has no solution. There is at least one in the opposite case.*

The first condition is due to Hilbert's *Nullstellensatz* theorem in the algebraically closed field $\mathbb{C}$: $1 \in \mathcal{I}_j$ is equivalent to $\mathcal{G}_j = \emptyset$. The existence of a solution needs the limit $\mathcal{G}_k$ of $(\mathcal{G}_j)$ be not empty. We have defined solutions as being differentiable functions of time. This has no meaning if the projection of $\mathcal{G}_k$ on the time axe is a discrete set. It corresponds to the second condition, since $\mathcal{G}_k \subset \mathcal{G}_j$ for $j \leq k$.

In the following we shall suppose both conditions are satisfied : talking about solutions in the neighborhood of regular points will have a concrete meaning.

In the neighbourhood of a regular point (t^*, y^*) of (1.1), we prove that there exists at least one differentiable function $y(t)$ such that $y(t^*) = y^*$ and (1.1) is satisfied. According to theorem 4.1, we can rewrite the system in the following way (it may need reordering the coordinates) :

$$(p_1, \ldots, p_{r_k})^T = b + S.(p_{r_k+1}, \ldots, p_n)^T$$

where $b \in \mathbb{C}(T, Y)^{r_k}$ and S is a $r_k \times u_k$-matrix having coefficients in $\mathbb{C}(T, Y)$. Since (t^*, y^*) is a regular point, neither b nor S have pole in some neighbourhood of this point. Nevertheless there are two possibilities :

- If $r_k = n$, the rewritten differential system satisfies the conditions for Cauchy-Lipschitz theorem. The right handside has continuous partial derivatives and no pole at (t^*, y^*). Therefore there exists a unique solution in some neighbourhood of (t^*, y^*).

- If $u_k > 0$, it is similar to the previous example : we define $y_{r_k+1}(t), \ldots, y_n(t)$ as arbitrary continuously differentiable functions of t, provided their value at t^* belongs to the projection of $\mathcal{G}_k$ on the space generated by the $r_{k+1}, \ldots, r_n$-th axes. Then the other derivatives are uniquely determined near (t^*, y^*). There are infinitely many solutions passing through this point.

4.3 Rectangular DAEs

In this paragraph we show that our definition of index extends to rectangular systems

$$A(T,Y)p = b(T,Y), \tag{4.3}$$

where A is a $m \times n$-matrix and b a vector of length m, both polynomials, and $m \neq n$.

1. **Subdetermined Case :** $m < n$.
   We add $n-m$ zero rows to A and augment b to a n-vector whose $n-m$ last components are zero. We have then a square $n \times n$ system, so our previous study can be applied, starting with $u_0 \geq n-m$. The index is well defined and if the existence conditions of lemma 4.1 are satisfied, there will be infinitely many solutions at regular points.

2. **Overdetermined Case :** $m > n$.
   The definition of relations matrices for the rows of A is still valid. Applying the iterative process of section 2, we have once again a decreasing sequence of algebraic varieties. The very difference with the square case (n=m) lies in the characterization of the index. In order to have no new equation underlying the system at j-th iteration, condition $r_j + u_j = n$ is not sufficient. Generally with $m > n$ we have $r_0 = n$ but this does not guarantee existence of solutions. The good condition of stationnarity is

   $$\mathcal{G}_j = \mathcal{G}_{j-1}$$

   so that the definition of the index keeps unchanged. Its existence once again results from the fact that $\mathbb{C}[T,Y]$ is a noetherian ring.

3. **Singular solutions** We can apply this last definition to the research of *singular solutions* of a quasi-linear ODE :

   $$F(t,y)dy/dt = g(t,y)$$

   where F is a polynomial regular matrix and $g(t,y)$ is a polynomial function. We suppose that $\det F$ is a nonzero polynomial, such that the equation

   $$\det F(t,y) = 0$$

   defines a hypersurface in $\mathbb{C}^{n+1}$. The singular solutions of the differential system above are the solutions of

   $$\left\{ \begin{array}{rcl} F(t,y)dy/dt & = & g(t,y) \\ 0 & = & \det F(t,y) \end{array} \right. . \tag{4.4}$$

   This is an overdetermined quasilinear DAE. Let's go further into investigations.

Equations (4.4) are generically $n-1$ differential ones and 2 algebraic ones. After differentiating these two constraints, we obtain a matrix A_1 of rank n and a new constraint. Its differentiation leads to a new constraint, since A_2 cannot have rank $> n$. So at each iteration we discover a new constraint. Generically it will stop when

$$\mathcal{G}_j = \emptyset,$$

which is the following

Corollary 4.1. *A generic quasi-linear ODE has no singular solution.*

Here is a 2-dimensional system with a complex parameter c :

$$\begin{cases} Y_1 p_1 + p_2 &= Y_2 + c \\ p_1 - p_2 &= Y_1 - c \end{cases}$$

The supplementary equation for singular solutions is the irreducible polynomial

$$1 + Y_1 = 0.$$

Substituting in the differential system we obtain

$$Y_2 + c = c - Y_1.$$

The first ideal is $\mathcal{I}_0 = (Y_1+1, Y_2-1)$. We differentiate and add the differential equations

$$p_1 = 0,\ p_2 = 0$$

so that the new constraint is

$$Y_1 - c = 0$$

and the second ideal is

$$\mathcal{I}_1 = (Y_1+1, Y_2-1, c+1).$$

If $c \neq -1$, $\mathcal{G}_1 = \emptyset$ so that there is no singular solution. Conversely, if $c=-1$, the unique singular solution is the constant function $t \mapsto (-1,1)$. We can conclude that for generic values of the parameter, there is no singular solution.

5 Extending to the reducible case

5.1 A general algorithm

To treat all square quasi-linear DAEs, we must take into account the reducible varieties, that is deciding if an ideal is primary or not. There are effective but expensive algorithms to answer this question (refer for example to Becker& Weispfenning's book [1]).

To end this paper we present an informal algorithm that summarizes the study of a general square quasi-linear DAE (see algorithm 5.1). Here are some comments about.

Algorithm 5.1. ALGEBRAIC DECOMPOSITION OF A POLYNOMIAL QUASI-LINEAR DAE.

***Inputs** : n integer ≥ 1, A a $n \times n$-matrix, b a n-vector, both having coefficients in $\mathbb{C}[T, Y]$.*

Preparation step

i:=0; ***Differential index***

$A_0 := A$; $b_0 := b$;

***Initialisation:** Compute the relations among the rows of A_0. Then $r_0, u_0, c_0, \mathcal{I}_0$ are known.*

Compute the prime ideals associated with $\mathcal{I}_0$: it provides the complete list of the irreducible components of $\mathcal{G}_0$.

For each irreducible component, calculate the rank r of A_0.

If $r = r_0$ then

Go to Main Loop

Else

$\mathcal{P}$ prime ideal of the component.

Replace (A_0, b_0) with (A_0, b_0) mod $\mathcal{P}$ and go to Initialisation.

Endif

Main Loop

$\mathcal{V}_i$ an irreducible component of $\mathcal{G}_i$

While $r_i + u_i < n$

$A_{i+1} := A_i$ bordered by the derivatives of the last discovered constraints.

$r_{i+1}, u_{i+1}, c_{i+1}, \mathcal{V}_{i+1} := \mathcal{G}_{i+1} \cap \mathcal{V}_i$ are computed.

Compute the prime ideals associated with $\mathcal{V}_{i+1}$: it provides the complete list of its irreducible components.

For each irreducible component, calculate the rank r.

If $r < r_{i+1}$ then

$\mathcal{P}$, prime ideal of the component.

Compute (A_{i+1}, b_{i+1}) mod $\mathcal{P}$ and start again the loop before increasing i.

Endif

i:=i+1;

Endwhile

End of the algorithm

***Outputs** : a list describing all the irreducible varieties where solutions lie.*
Final matrix, constraints equations and index for each component.

- **Preparation Step** : we use the method described in section 2 to compute the variety $\mathcal{G}_0$. Linear algebra tools in $\mathbb{C}[T,Y]$ allow us to find a relations matrix of A and Gröbner basis technics provide a set of equations of $\mathcal{G}_0$.

  When $\mathcal{G}_0$ is not irreducible, $\mathcal{I}_0 = \bigcap_j \mathcal{P}_j$ where the $\mathcal{P}_j$'s are primary ideals pairwise not related to the same prime ideals. Hence

  $$\mathcal{G}_0 = \bigcup_j \mathcal{V}_j$$

  where each $\mathcal{V}_j$ is an irreducible algebraic variety. So if for some j, $\mathcal{V}_j$ is a subset of the variety

  $$\{(t,y) \in \mathbb{C}^{n+1} \ / \ \mathrm{rank} A(t,y) < \mathrm{rank} A\}$$

  there's at least one new algebraic constraint on $\mathcal{V}_j$: we must put

  it in the system before adding the equations of the tangent space to the system. This leads to a subvariety of $\mathcal{V}_j$, where we do the same verification on rank until stabilization.

- **Main Loop** : if the index is greater than 1, we enter this loop until we find a system of maximal rank. If we encounter a reducible variety, we make a branching on each of its irreducible components.

To make the algorithm clearer we treat the following example :

$$\left(\begin{array}{ccc|c} Y_3 & Y_1-2 & Y_2^2 & -1 \\ 0 & Y_1-2 & Y_2^2 & Y_1 - T \\ 0 & 0 & 0 & Y_3(Y_1^2 - 3Y_3Y_2 - 4) \end{array}\right)$$

Here $\mathcal{G}_0 = {\mathcal{V}_0}^1 \cup {\mathcal{V}_0}^2$ where

$${\mathcal{V}_0}^1 = \{Y_3 = 0\}$$

and

$$\mathcal{V}_0^2 = \{Y_1^2 - 3Y_2Y_3 = 4\}.$$

- **Study on $\mathcal{V}_0^1$** : the rank of A_0 is 2 in $\mathbb{C}^4$, but computing it in $\mathcal{V}_0^1$ we get the value $r_0 = 1$, and the system modulo (Y_3) is

  $$\left(\begin{array}{ccc|c} 0 & Y_1-2 & Y_2^2 & -1 \\ 0 & Y_1-2 & Y_2^2 & Y_1 - T \end{array}\right).$$

  There is a new constraint,

  $$Y_1 - T + 1 = 0,$$

  so that $c_0 = 2$ for the component $\mathcal{V}_0^1$. We actually have to study the system on its subvariety $\mathcal{W}_0 = \{Y_3 = 0\} \cap \{Y_1 = T - 1\}$: after differentiating we find

  $$\left(\begin{array}{ccc|c} 0 & Y_1-2 & Y_2^2 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The determinant of A_1 is $2 - Y_1$ which does not vanish at most points of $\mathcal{W}_0$. It means $r_1 = 3$ and the index is 1.

- **Study on** $\mathcal{V}_0^2$: it is much simpler since $r_0 = 2$ in this component. We differentiate the constraint to get

$$\left(\begin{array}{ccc|c} Y_3 & Y_1 - 2 & Y_2^2 & -1 \\ 0 & Y_1 - 2 & Y_2^2 & Y_1 - T \\ 2Y_1 & -3Y_3 & -3Y_2 & 0 \end{array} \right).$$

the determinant is

$$(Y_1^2 - 4)(2 - Y_1 + Y_2 Y_3)$$

and $r_1 = 3$: the index is 1 in $\mathcal{V}_0^2$.

In other examples we may have change of index depending on the components, or loss of uniqueness on a component.

5.2 Conclusion

In the quasi-linear case we have given an intrinsic definition of the index of a DAE, based on a geometric approach. We have showed that constraints varieties of polynomial such DAEs are always algebraic, so that they are computable in practice. Then we have observed the index is an invariant of each irreducible components of the last constraint variety.

Knowing where the initial conditions have to be located, we could define what are regular points. At such points the study of existence and uniqueness of a differentiable function satisfying the DAE is derived simply, since there is no problem in inverting the linear system for the derivatives.

The algorithm we obtained is a practical translation of Reich-Rabier-Rheinboldt method [12] [11] or LeVey's ideas [15]. We made the results more precise for quasi-linear systems, which are the most often encountered in applications. In a forthcoming paper we shall look more carefully at fixed singular points, where theorems of existence and uniqueness fail. We hope this symbolic approach will be useful in the handling of DAEs ; our work may be seen as a preliminary step, before numerical integration or studying the behaviour of solutions near singularities for examples.

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