A Robust Control System Design by a Special Quantifier Elimination Methods using a Sturm-Habicht sequence

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Abstract

For multi-objective design and robust control problems, the methods based on Quantifier Elimination (QE) have been proposed. QE based methods are really suitable for such problems but, in general, they have a drawback on computational complexity. A parameter space approach for robust control system design was developed by reducing important design specifications to Sign Definite Condition (SDC) by S.Hara *et.al.*. They proposed to check SDC by utilizing D.Šiljak's criterion. This also gives an efficient QE algorithm for particular inputs SDC. In this paper, we propose a robust control system design method based on more efficient special QE algorithm for SDC using *Sturm-Habicht sequence*. This method has better worst-case complexity than that using D.Šiljak's criterion and a nice specialization property.

1 Introduction

Multi-objective design and robust control synthesis are of great practical interest and main concerns in the control system design. However, in general, they are hard to solve and there are no analytical solutions. Recently, for such problems, the methods based on Quantifier Elimination (QE) were proposed by several researchers (see [5][12][15][1]).

QE deals with the *first-order formulas*, which consists of polynomial equations, inequalities, quantifiers ($\forall, \exists$) and Boolean operators ($\wedge, \vee, \neg, \rightarrow, etc$). QE procedure is an algorithm to compute equivalent quantifier-free formula for a given first-order formula over the real closed field. For the input $\forall x(x^2 + bx + c > 0)$, QE outputs the equivalent quantifier-free formula; $b^2 - 4c < 0$. (As for QE see [2])

For example, in [5] it is shown that how certain robust multi-objective design problems can be reduced to QE problems and actually solved by using "QEPCAD". QEPCAD is a symbolic computation package for QE based on the Cylindrical Algebraic Decomposition (CAD) algorithm presented by G.E.Collins [3]. In [12] it is shown that, in feedback design of linear time-invariant systems, robustness and several performance specifications (H_∞ norm constraint, gain and phase margins) on the close-loop system can also be solved as QE problems by using QEPCAD.

In this article, we consider this kind of problem, in particular, focus on a robust control system design methods based on QE. QE based approach is really effective for such problems. However,

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unfortunately the size of the problems which can be solved by QE based approach is limited, because the computational complexity of the general QE algorithm based on CAD algorithm is doubly exponential in the number of quantified variables (including parameter variables).

In applications of QE to control problems so far, QE method is applied to the first-order formulas derived from the control problems by a direct translation. For the efficient computation, it is important to reduce the target problems to a first-order formula as simple as possible. Furthermore, it is preferable to use special QE algorithm which is effective for a particular input. Hence, we should try to translate the control system design problem to a formula to which a special QE algorithm is applicable. As one of such formulas, there is a "*Sign Definite Condition (SDC)*" for robust control system design problems.

A parameter space design method is known to be one of the useful tools to deal with multi-objective design problems. A parameter space approach for robust control system design is developed by reducing important design specifications such as H_∞ norm constraint, stability margins *etc*, which are frequently used as indices of the robustness, to sign definite condition. See [11][13][14]. The sign definite condition is a very simple (first-order) formula and suited for a QE procedure in view of computational efficiency. Moreover, In [11] it is also proposed that SDC is checked by using Routh-Hurwitz like criterion proposed by D.Šiljak for positive realness [17]. A parameter space approach based on SDC using D.Šiljak's criterion is essentially equivalent to performing QE for the particular inputs

$$\forall x > 0,\ f(x) > 0 \tag{1}$$

where $f(x)$ is a polynomial with real coefficients. So this method is regarded as a special QE algorithm for the particular input first-order formula (1) and more efficient than the general QE algorithm based on CAD algorithm. However in the method using D.Šiljak's criterion, there remains some issues related to singular cases (see [6]) and specialization of parameters.

Hence, in this paper, we propose a parameter space approach for robust control system design based on a special QE method for SDC using Sturm-Habicht sequence. A combinatorial algorithm to solve the particular QE problem $\forall x,\ f(x) > 0$ based on Sturm-Habicht sequence is proposed by L.González-Vega *et.al.*[8]. We utilize their algorithm with some modification for a sign definite condition (1). The method proposed here is more efficient than the method using Routh-Hurwitz like criterion by D.Šiljak and moreover has a good specialization property.

2 Sign definite condition (SDC)

In this paper we use $\mathbf{R}$ and $\mathbf{Q}$ for the fields of real numbers and rational numbers, respectively.

Definition 1 *Let $f(x)$ be a polynomial in x over $\mathbf{R}$ i.e. $f(x) \in \mathbf{R}[x]$. $f(x)$ is sign definite in the interval $x \in [a, b]$ such that $a < b$ $(\in \mathbf{R})$, denoted by $f(x) \in \mathbf{N}_0[a, b]$, if $f(x)$ preserves its sign in $[a, b]$, or does not cross zero in $[a, b]$.*

Note that in actual computation we consider the polynomial $f(x)$ over $\mathbf{Q}$. This restriction is needed since we utilize a computer algebra system. In this paper we, in particular, consider the parametric case that is the coefficients of $f(x)$ contain some real parameters, say, $p_1, \cdots, p_s$. Strictly speaking, this means $f(x)$ is a polynomial over the rational function fields $\mathbf{R}(p_1, \cdots, p_s)$ *i.e.* $f(x) \in \mathbf{R}(p_1, \cdots, p_s)[x]$. Without loss of generality, it is enough to consider the problem

$$f(x) \in \mathbf{N}_0[\ 0, +\infty\ ], \tag{2}$$

because the condition $f(x) \in \mathbf{N}_0[a, b]$ can be translated to the condition $f(z) \in \mathbf{N}_0[0, +\infty]$ by a bilinear transformation $z = -\frac{x-a}{x-b}$.

2.1 Reduction of the robustness specifications to SDC

The sign definition condition have emerged as the important problem in a parameter space approach for robust control system design. The specifications such as

- H_∞ norm constraint,
- frequency restricted norm constraint,
- gain and phase margin constraint, and
- pole location,

that are frequently used as indices of robustness of feedback control systems, are reduced to sign definite condition (see [11][13][14][17]). This fact makes it appealing to look into the SDC.

2.2 Checking SDC by Šiljak's criterion

In [11][13][14], it is shown that SDC can be readily checked by the following lemma based on the Routh-Hurwitz like criterion proposed by D.Šiljak [17]:

Lemma 1 *[13] Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbf{R}[x]$. $f(x)$ is sign definite in $x \in [0, +\infty]$ if and only if*

$$V[f(x)] = n$$

holds, where V is the number of sign changes of the most left column of the Modified Routh Array defined by

$$\begin{array}{ccccc} (-1)^n a_n & (-1)^{n-1} a_{n-1} & \cdots & -a_1 & a_0 \\ (-1)^n n a_n & (-1)^{n-1}(n-1) a_{n-1} & \cdots & -a_1 & \\ \vdots & & & & \\ a_0 & & & & \end{array}$$

The first two rows of Routh array above are formed by the coefficients of the polynomial $f(-x)$ and $f'(-x)$, and following rows in Routh array are formed by the coefficients of the polynomial remainder sequence generated by Euclidean divisions. Here we enumerate the issues when we use the Routh type criterion.

- In the computation of the remainder sequence by using exact arithmetic, the size of the (rational) coefficients of the polynomials appearing in the sequence grows exponentially in the degree of the polynomial.
- In the case where the coefficients contain some parameters, there remains the problem concerning specialization of parameters; Since rational functions may appear in the sequence due to Euclidean division procedure, "division by 0" may occur by substitution of parameters by real numbers. Then we have to recompute completely for the special values of the parameters. (see an example in §6).
- Moreover, separately from the regular case, we have to take care the singular cases which occur when (i) an element of the first column become zero (not all the elements in the corresponding row are zero), (ii) all the elements in a row of the array vanish simultaneously. (See Remark 4 in [17], or [6] for details.)

3 Sturm-Habicht sequence

Here we show the definition of Sturm-Habicht sequence and mention its algebraic properties briefly (see [9][10][8] for details). Sturm-Habicht sequence can be used for real root counting as is Sturm sequence. Firstly, we introduce a subresultant. (Note that the following definition is slightly different from the ordinary one [7]):

Definition 2 *[10] Let P, Q be polynomials in $\mathbf{Q}[x]$*

$$P = \sum_{k=0}^{n} a_k x^k \quad Q = \sum_{k=0}^{m} b_k x^k$$

where n, m be non-negative integers and let $\ell = min(n, m)$. For $i = 0, 1, \cdots, \ell$ we define the subresultant associated to P, n, Q and m of index i as follows:

$$Sres_i(P, n, Q, m) = \sum_{j=0}^{i} M_j^i(P, Q) x^j$$

where $M_j^i(P, Q)$ is the determinant of the matrix composed by the columns $1, 2, \cdots, n + m - 2i - 1$ and $n + m - i - j$ in the matrix $s_i(P, n, Q, m)$:

$$s_i = \begin{array}{c} \overbrace{\hphantom{a_n \cdots a_0 \quad \cdots \quad a_0}}^{n+m-i} \\ \begin{pmatrix} a_n & \cdots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_n & \cdots & a_0 \\ b_n & \cdots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_n & \cdots & b_0 \end{pmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m-1 \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-1 \end{array}$$

Now Sturm-Habicht sequence associated to P, Q is defined as the subresultant sequence for P and $P'Q$ with certain sign changes.

Definition 3 *[10] Let P, Q be polynomials in $\mathbf{Q}[x]$ with degrees n, m, respectively. Here n, m be non-negative integers. Let $v = n + m - 1$ and*

$$\delta_k = (-1)^{\frac{k(k+1)}{2}}$$

for every integer k. The Sturm-Habicht sequence associated to P and Q is defined as the list of the polynomials $\{SH_j(P, Q)\}_{j=0,\cdots,v+1}$ given by

- *$SH_{v+1}(P,\ Q) = P$,*
- *$SH_{v+1}(P, Q) = P'Q$, and*
- *$SH_j(P, Q) = \delta_{v-j} Sres_j(P, v + 1, P'Q, v)$*

for every $j = 0, 1, \cdots, v - 1$ where $P' = \frac{dP}{dx}$. For every j the principal j-th Sturm-Habicht coefficient is defined by the coefficient of degree j of $SH_j(P, Q)$ (denoted by $st_j(P, Q)$) and the constant term of $SH_j(P, Q)$ is denoted by $ct_j(P, Q)$.

Then, we move to the relation between the number of real roots of a polynomial and the Sturm-Habicht sequence. We introduce the following integer for every $\epsilon \in \{-1, 0, 1\}$ and $\alpha, \beta \in \mathbf{R} \cup \{-\infty, +\infty\}$ with $\alpha < \beta$:

$$c_\epsilon^{[\alpha,\beta]}(P; Q) = \#\{\gamma \in (\alpha, \beta) \mid P(\gamma) = 0,\ sign(Q(\gamma)) = \epsilon\}.$$

Now we consider how to compute

$$c_+^{[\alpha,\beta]}(P; Q) - c_-^{[\alpha,\beta]}(P; Q). \tag{3}$$

In the case where $Q = 1$, this gives the number of real zeros of P in $[\alpha, \beta]$.

Now we prepare several functions for sign counting which we need to compute the integer (3).

Definition 4 *Let $\{a_0, a_1, \cdots, a_t\}$ be a list of non zero elements in $\mathbf{R}$, where t is a positive integer. We define $V(\{a_0, a_1, \cdots, a_t\})$ as the number of sign variations in the list $\{a_0, a_1, \cdots, a_t\}$.*

Definition 5 *Let $P, Q \in \mathbf{Q}[x]$, α an element of $\mathbf{R} \cup \{-\infty, +\infty\}$ such that $P(\alpha) \neq 0$ and $\{SH_j(P,Q)\}_{j=0,\cdots,v+1}$ the Sturm-Habicht sequence associated to P and Q. Let $\{g_0, \cdots, g_s\}$ be a list of polynomials obtained from $\{SH_j(P,Q)\}_{j=0,\cdots,v+1}$ by discarding the polynomials identically 0.*

We define $W_{SH}(P, Q; \alpha)$ as the number of sign variations in the list $\{g_0(\alpha), \cdots, g_s(\alpha)\}$ with following rule:

- *count 1 change for* $[-0+][+0-][+00-][-00+]$
- *count 2 changes for* $[+00+][-00-]$

If not in these two cases, use ordinary definition for sign change.

Note that Theorem 2 implies that it is not possible to find more than two consecutive zeros in the sequence $\{g_0(\alpha), \cdots, g_s(\alpha)\}$ and $[+0+][-0-]$ cannot appear. We denote the number $W_{SH}(P, Q; \alpha) - W_{SH}(P, Q; \beta)$ by $W_{SH}(P, Q; \alpha, \beta)$. Then we have the following theorem :

Theorem 1 *[10] Let $P, Q \in \mathbf{Q}[x]$ and $\alpha, \beta \in \mathbf{R} \cup \{-\infty, +\infty\}$ s.t. $\alpha < \beta$. Then*

$$W_{SH}(P, Q; \alpha, \beta) = c_+^{[\alpha,\beta]}(P; Q) - c_-^{[\alpha,\beta]}(P; Q).$$

From this theorem, the number of real zeros of P in $[0, +\infty]$ *i.e.* $c_+^{[0,+\infty]}(P; 1) - c_-^{[0,+\infty]}(P; 1)$ is determined by counting the number $W_{SH}(P, 1; 0, +\infty)$.

4 Algorithm

Now we present a robust control system design method based on a more efficient special QE algorithm for SDC using Sturm-Habicht sequence. Usage of Sturm-Habicht sequence

- resolves the exponential growth of coefficients,
- clears away the specialization problem,
- makes us free from the care for singular cases (*i.e.* we can deal with all the cases uniformly),

by virtue of subresultants instead of remainders by Euclidean divisions. Furthermore, Sturm-Habicht sequence has good worst-case computational complexity (See [9] for details.)

4.1 Sturm-Habicht sequence computation

Let $f(x) \in \mathbf{R}[x]$ with degree n. Sturm-Habicht sequence of a polynomial $f(x)$ is defined as the *subresultant* sequence starting from $f(x)$ and $f'(x)$ modulo some specified sign changes. We have the following theorem [9]:

Theorem 2 (Structure theorem) *For every $k \in \{0, 1, \cdots, n-1\}$, let $H_k = SH_k(f)$ and $h_k = st_k(f)$ for short. And let $h_n = 1$. Then for every $j \in \{1, \cdots, n-1\}$ such that $h_{j+1} \neq 0$ and $deg(H_j) = r \leq j$, we have*

1. *if $r < j - 1$ then $H_{j-1} = \cdots = H_{r-1} = 0$,*
2. *if $r < j$ then $h_{j+1}^{j-r} H_r = \delta_{j-r} LC(H_j)^{j-r} H_j$,*
3. *$h_{j+1}^{j-r+2} H_{r-1} = \delta_{j-r+2} Prem(H_{j+1}, H_j)$.*

where $LC(A)$ stands for the leading coefficient of a polynomial A and $Prem(A, B)$ is a remainder obtained by division of $LC(B)^{n-m+1} A$ by B for polynomials A, B with degree n, m, respectively.

Sturm-Habicht sequence of a polynomial f is constructed according this theorem and then we need $O(n^2)$ algebraic operations in $\mathbf{Q}(p_1, \cdots, p_s)$.

4.2 Checking SDC

Let the Sturm-Habicht sequence of f be $\{SH_j(f)\}_{\ j=0,\cdots,n} = \{g_0, \cdots, g_s\}$. Then for $\alpha \in \mathbf{R} \cup \{-\infty, +\infty\}$ we define $W_{SH}(f;\alpha)$ as the number of sign variations in the list $\{g_0(\alpha), \cdots, g_s(\alpha)\}$. And let $W_{SH}(f;\alpha,\beta) = W_{SH}(f;\alpha) - W_{SH}(f;\beta)$. For every j, the principal j-th Sturm-Habicht coefficient is defined as the coefficient of degree j of $SH_j(f)$. We denote the principal j-th Sturm-Habicht coefficient by $st_j(f)$ and the constant term of $SH_j(f)$ by $ct_j(f)$.

The sign definiteness of f in the interval $[0, +\infty]$ is equivalent to that f has no real roots in $[0, +\infty]$. Hence, an equivalent condition to the sign definition condition in $[0, +\infty]$ is obtained according to the following proposition (*cf.* Theorem 1).

Proposition 1 *A polynomial $f(x)$ is sign definite in $[0, +\infty]$ if and only if $W_{SH}(f; 0, +\infty) = 0$.*

By definitions we have

$$W_{SH}(f;0,+\infty) = W_{SH}(f;0) - W_{SH}(f;+\infty) = V(\{ct_n(f), \cdots, ct_0(f)\}) - V(\{st_n(f), \cdots, st_0(f)\}) \quad (*)$$

The last formula $(*)$ gives us how we count the number $W_{SH}(f;0,+\infty)$ concretely. Since $ct_0(f) = st_0(f)$, we need only $2(n+1) - 1 = 2n+1$ sign evaluations.

If we have Sturm-Habicht sequence for f, we construct the (quantifier-free) equivalent condition for SDC of f by the following procedure. The obtained conditions are of the form of the union of semi-algebraic sets.

1. consider all the 3^{2n+1} (at most) possible sign conditions over $ct_i(f)$'s and $st_i\ (f)$'s,
2. choose all sign conditions which satisfy $W_{SH}(f;0) - W_{SH}(f;+\infty) = 0$ according to $(*)$,
3. construct semi-algebraic sets generated by $ct_i(f)$'s and $st_i(f)$'s for each selected sign conditions and combine them as a union.

Remark 1 *Once we execute this algorithm for the generic polynomial with degree n*

$$F_n(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

the result can be used for any other polynomials with degree n by substituting the coefficients c_i by those of an input polynomial. (In the case of F_2, see a example in §6.) So, the results for the generic cases should be stored in a database (or table) to be called upon, whenever needed. This greatly improves the total efficiency of our methods.

4.3 Simplification

The result through above procedure obviously tends to be large and complicated, and hence we should reduce the result as simple as possible. Some possible simplifications are as follows:

- Manual simplifications by deleting some sign conditions trivially false (*i.e.* empty) or decreasing the number of unions by using the well-known rules;

  $$< \cup > \to \neq, \ < \cup = \to \leq, \ > \cup = \to \geq$$

  are indicated in [8].

- We, fortunately, have some sophisticated softwares for automatic formula simplification which are implemented on a QE package "REDLOG" ‡ and another QE package on a computer algebra system "Risa/Asir" ¶.

‡REDLOG is developed at University of Passau (Germany) on a computer algebra system REDUCE, see [4]. It is based on the virtual substitution method of parametric test points proposed by V. Weispfenning [18].

¶Risa/Asir is developed at Fujitsu labs, see [16], anonymous -ftp via: `endeavor.fujitsu.co.jp:/pub/isis/asir`. Both CAD algorithm and virtual substitution method are implemented on Risa/Asir.

5 Example

Here we demonstrate our method by applying it two examples. All the computations were done by using a computer algebra system Risa/Asir and the results were all obtained immediately on a PC with Pentium 200MHz CPU.

Example 1 (sensitivity analysis of a PI control system) : We consider a PI control system shown in Fig.1. The structure of the compensator is fixed as $C(s) = k + \frac{m}{s}$.

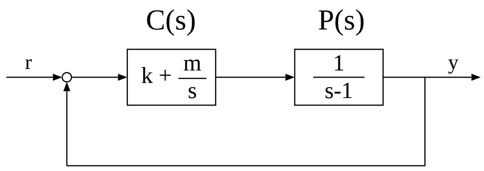

Figure 1: PI control system

The sensitivity and complementary sensitivity functions are given by

$$S(s) = \frac{1}{1 + P(s)C(s)} = \frac{s^2 - s}{s^2 + (k-1)s + m} \tag{4}$$

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{ks + m}{s^2 + (k-1)s + m} \tag{5}$$

The goal is to determine the possible range of the parameters k and m which satisfy the specifications

$$||S(s)||_{[0,\omega_s]} < \gamma_s, \tag{6}$$

$$||T(s)||_{[\omega_t,\infty]} < \gamma_t, \tag{7}$$

where $||G||_{[\omega_1,\omega_2]}$ is a norm defined for a restricted frequency domain $[\omega_1, \omega_2]$ *i.e.*

$$||G||_{[\omega_1,\omega_2]} = \sup_{\omega_1 \le \omega \le \omega_2} \bar{\sigma}(G(j\omega))$$

if we denote the maximal singular value of G by $\bar{\sigma}(G(j\omega))$. As shown in [14], the both specifications (6) and (7) are reduced to the sign definite conditions. The specification (6) is equivalent to the following SDC:

$$f_s(x) = a_2x^2 + a_1x + a_0 \ \in \mathbf{N}_0[0, +\infty] \tag{8}$$

where

$$\begin{aligned} a_2 &= -\omega_s^4 - (2m\alpha + (1 - k\alpha)^2)\omega_s^2 + m^2\alpha, \\ a_1 &= (2m\alpha + (1 - k\alpha)^2)\omega_s^2 - 2m^2\alpha, \\ a_0 &= m^2\alpha \end{aligned}$$

with $\alpha = \frac{\gamma_s^2}{(1-\gamma_s^2)}$. And the specification (7) is equivalent to the following:

$$f_t(x) = b_2x^2 + b_1x + b_0 \ \in \mathbf{N}_0[0, +\infty] \tag{9}$$

where

$$\begin{aligned} b_2 &= 1, \\ b_1 &= -2\omega_t^2 + 2m - (1-k)^2 + \frac{k^2}{\gamma_t^2}, \\ a_0 &= \omega_t^4 - (2m - (1-k)^2 + \frac{k^2}{\gamma_t^2})\omega_t^2 + m^2(1 - \frac{1}{\gamma_t^2}). \end{aligned}$$

Consequently, what we do to obtain the possible range of k, m such that (8),(9) is determining the SDC for the generic polynomial with degree 2:

$$F_2(x) = c_2x^2 + c_1x + c_0 \ \in \mathbf{N}_0[0, +\infty]$$

Sturm-Habicht sequence $\{SH_j(F_2)\}_{j=2,1,0}$ of $F_2(x)$ consists of

$$\begin{array}{rcl} SH_2(F_2) & = & c_2x^2 + c_1x + c_0, \\ SH_1(F_2) & = & 2c_2x + c_1, \\ SH_0(F_2) & = & c_2c_1^2 - 4c_0c_2^2. \end{array}$$

Then immediately we have

$$\begin{array}{rcl} \{ct_i\}_{i=2,1,0} & = & \{c_0, c_1, (c_2c_1^2 - 4c_0c_2^2)\} \\ \{st_i\}_{i=2,1,0} & = & \{c_2, 2c_2, (c_2c_1^2 - 4c_0c_2^2)\} \end{array}$$

Hence we check whether the number $W_{SH}(F_2; 0, +\infty)$ is equal to 0 or not according to the formula $(*)$ for 3^4 sign conditions $\{-, 0, +\}^4$ over the sequence

$$\{c_0, c_1, c_2, (c_2c_1^2 - 4c_0c_2^2)\}.$$

Finally, in the case of (8), we reaches the results; $f_s(x) \in \mathbf{N}_0[0, +\infty]$ if and only if

$$\begin{array}{c} [\ a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge (a_2a_1^2 - 4a_0a_2^2) > 0\ ] \cup \\ [\ a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge (a_2a_1^2 - 4a_0a_2^2) < 0\ ] \cup \\ \vdots \\ \cup\ [\ a_0 < 0 \wedge a_1 < 0 \wedge a_2 < 0 \wedge (a_2a_1^2 - 4a_0a_2^2) < 0\ ]. \end{array}$$

If we substitute ω_s, γ_s with the appropriate values, then this result gives the possible range of the parameters k, m as a union of the semi-algebraic sets. The possible range is visualized easily by plotting the semi-algebraic sets on $k - m$ plane.

In the case of (9), since $c_2 = b_2 = 1$ we check whether the number $W_{SH}(f_t; 0, +\infty) = 0$ or not for 3^3 sign conditions over the sequence $\{c_0, c_1, (c_1^2 - 4c_0)\}$.

Remark 2 *For a certain class of plants with structured uncertainties, robust performance problem can be reduced to SDC by utilizing Kharitonov's theorem (see [11][13]) and hence is solved by the method proposed in §3. For example, in [11][13], it is shown that for the same PI control system as in Fig.1 with a plant with structured uncertainties, norm constraints can be decomposed to a conjunction of SDCs and the stability margins constraint is satisfied if and only if the the Kharitonov systems associated to the open loop system satisfy the constraints.*

Remark 3 *Note that no rational polynomials appears in Sturm-Habicht sequence. On the other hand, modified Routh array of $F_2(x)$ is given as follows:*

$$\begin{array}{ccc} c_2 & -c_1 & c_0 \\ 2c_1 & -c_0 & \\ \frac{-c_0c_2+2c_1^2}{2c_1} & c_0 & \\ \frac{c_2c_0^2-6c_1^2c_0}{-c_2c_0+2c_1^2} & & \\ c_0 & & \end{array}$$

In the most left column, rational functions appear due to Euclidean division. This leads to bad specialization property i.e. "division by 0" by specialization. For example, if $c_1 = 0$, the denominator of $\frac{-c_0c_2+2c_1^2}{2c_1}$ vanishes and specialization is impossible.

Example 2 (a generic quartic polynomial F_4) This is the first non-trivial case.

$$F_4 = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0.$$

Determine the SDC for the generic polynomial with degree 4. Sturm-Habicht sequence $\{SH_j(F_4)\}_{j=4,3,2,1,0}$ of $F_2(x)$ are given as

$$\begin{array}{rcl} SH_4(F_4) &=& c_4x^4+c_3x^3+c_2x^2+c_1x+c_0, \\ SH_3(F_4) &=& 4c_4x^3+3c_3x^2+2c_2x+c_1, \\ SH_2(F_4) &=& (-8c_4^2c_2+3c_4c_3^2)x^2+(-12c_4^2c_1+2c_4c_3c_2)x+c_4c_3c_1-16c_0c_4^2, \\ SH_1(F_4) &=& (-36c_4^3c_1^2+(28c_4^2c_3c_2-6c_4c_3^3)c_1-8c_4^2c_2^3+2c_4c_3^2c_2^2+32c_0c_4^3c_2-12c_0c_4^2c_3^2)x \\ &=& +3c_4^2c_3c_1^2+(-4c_4^2c_2^2+c_4c_3^2c_2-48c_0c_4^3)c_1+32c_0c_4^2c_3c_2-9c_0c_4c_3^3, \\ SH_0(F_4) &=& -27c_4^3c_1^4+(18c_4^2c_3c_2-4c_4c_3^3)c_1^3+(-4c_4^2c_2^3+c_4c_3^2c_2^2+144c_0c_4^3c_2 \\ && -6c_0c_4^2c_3^2)c_1^2+(-80c_0c_4^2c_3c_2^2+18c_0c_4c_3^3c_2-192c_0^2c_4^3c_3)c_1 \\ && +16c_0c_4^2c_2^4-4c_0c_4c_3^2c_2^3-128c_0^2c_4^3c_2^2+144c_0^2c_4^2c_3^2c_2-27c_0^2c_4c_3^4+256c_0^3c_4^4. \end{array}$$

Then we have

$$\begin{array}{rcl} \{st_i\}_{i=4,3,2,1,0} &=& \{c_4, 4c_4, -8c_4^2c_2+3c_4c_3^2, \\ && -36c_4^3c_1^2+(28c_4^2c_3c_2-6c_4c_3^3)c_1-8c_4^2c_2^3+2c_4c_3^2c_2^2+32c_0c_4^3c_2-12c_0c_4^2c_3^2, SH_0(F_4)\} \\ \{ct_i\}_{i=4,3,2,1,0} &=& \{c_0, c_1, c_4c_3c_1-16c_0c_4^2, \\ && 3c_4^2c_3c_1^2+(-4c_4^2c_2^2+c_4c_3^2c_2-48c_0c_4^3)c_1+32c_0c_4^2c_3c_2-9c_0c_4c_3^3, SH_0(F_4)\} \end{array}$$

Hence we consider the set of all sign conditions $\{\varepsilon_7,\cdots,\varepsilon_0\}$ $(\varepsilon_i \in \{-,0,+\})$ for $\{S_7,\cdots,S_0\}$ which satisfy that the number $W_{SH}(F_4;0,+\infty)$ is equal to 0. Here S_i's are given by

$$\begin{array}{rcl} S_7 &=& c_4, \\ S_6 &=& 3c_3^2c_4-8c_2c_4^2 \quad = c_4(3c_3^2-8c_2c_4), \\ S_5 &=& -36c_4^3c_1^2+(28c_2c_3c_4^2-6c_3^3c_4)c_1+32c_0c_2c_4^3+(-12c_0c_3^2-8c_2^3)c_4^2+2c_2^2c_3^2c_4, \\ &=& 2c_4(-18c_4^2c_1^2+(14c_2c_3c_4-3c_3^3)c_1+16c_0c_2c_4^2+(-6c_0c_3^2-4c_2^3)c_4+c_2^2c_3^2) \\ S_4 &=& SH_0(F_4) \qquad = c_4\cdot S_0', \\ S_3 &=& c_0, \\ S_2 &=& c_1, \\ S_1 &=& c_3c_4c_1-16c_0c_4^2 \;= c_4(c_3c_1-16c_0c_4), \\ S_0 &=& 3c_3c_4^2c_1^2+(-48c_0c_4^3-4c_2^2c_4^2+c_2c_3^2c_4)c_1+32c_0c_2c_3c_4^2-9c_0c_3^3c_4, \\ &=& c_4(3c_3c_4c_1^2+(-48c_0c_4^2-4c_2^2c_4+c_2c_3^2)c_1+32c_0c_2c_3c_4-9c_0c_3^3). \end{array}$$

where

$$\begin{array}{l} S_0' = -27c_4^2c_1^4+(18c_2c_3c_4-4c_3^3)c_1^3+(144c_0c_2c_4^2+(-6c_0c_3^2-4c_2^3)c_4+c_2^2c_3^2)c_1^2+(-192c_0^2c_3c_4^2 \\ -80c_0c_2^2c_3c_4+18c_0c_2c_3^3)c_1+256c_0^3c_4^3-128c_0^2c_2^2c_4^2+(144c_0^2c_2c_3^2+16c_0c_2^4)c_4-27c_0^2c_3^4-4c_0c_2^3c_3^2 \end{array}$$

Note that $\varepsilon_3 \in \{+,-\}$ since here we assume that $c_0 \neq 0$. Moreover from the algebraic properties of Sturm-Habicht sequence it it impossible that more than two consecutive zeros appear in the sequence. This also implies that $c_4 \neq 0$. Then we have the necessary and sufficient conditions, the union of the following 561 semialgebraic sets, such that F_4 satisfies sign definite condition :

$$\begin{array}{c} [\ S_7>0 \ \wedge\ S_6>0 \ \wedge\ S_5>0 \ \wedge\ S_4>0 \ \wedge\ S_3>0 \ \wedge\ S_2>0 \ \wedge\ S_1>0 \ \wedge\ S_0>0\ ] \cap \\ [\ S_7>0 \ \wedge\ S_6<0 \ \wedge\ S_5>0 \ \wedge\ S_4>0 \ \wedge\ S_3>0 \ \wedge\ S_2>0 \ \wedge\ S_1<0 \ \wedge\ S_0>0\ ] \cap \\ \vdots \\ [\ S_7<0 \ \wedge\ S_6<0 \ \wedge\ S_5<0 \ \wedge\ S_4<0 \ \wedge\ S_3<0 \ \wedge\ S_2<0 \ \wedge\ S_1<0 \ \wedge\ S_0<0\ ]. \end{array}$$

Furthermore, this formula can be simplified by deleting trivially empty semialgebraic sets based on the followings:

- $S_1=0,\ \ S_2=0 \implies c_1=0, c_0=0 \implies$ This is contrary to $c_0 \neq 0$.
- $S_6=0 \implies c_2=\frac{3c_3^2}{8c_4} \implies S_5=-(16c_4^2c_1-c_3^3)^2 \leq 0$
- $S_1=0 \implies c_3=\frac{16c_0c_4}{c_1} \implies S_0=\frac{-c_4^2(c_2c_1^2-96c_0^2c_4)^2}{c_1^3} \implies S_0, S_2$ have different sign.

Finally we have the union of the 477 semialgebraic sets, such that F_4 satisfies the sign definite condition (Total computation time on Risa/Asir is 65.26 seconds).

6 Conclusion

In this paper, we have presented a new parameter space approach for robust control system design based on a special QE method for SDC using Sturm-Habicht sequence. This method is more efficient than the method using Routh-Hurwitz like criterion and has a good specialization property.

We have demonstrated a simple example in §6. We should examine the practical efficiency of our method by further experiment on a computer algebra system especially for the actual problems.

The proposed method is regarded as a special QE algorithm for the particular input formula (1). The extension of this direction to general QE method is proposed by V. Weispfenning [19].

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