Quantifier Elimination for Trigonometric Polynomials by Cylindrical Trigonometric Decomposition

Extended Abstract

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1. Introduction

The quantifier elimination (shortly QE) problem is the following: *Given a formula with polynomial equations and inequalities and quantifiers, construct an equivalent formula (i.e. a formula with the same solution set over* **R***) without quantifiers.* It has been observed that a broad range of problems in mathematics, computer science, engineering and industrial computations can be reduced to the QE problem (see (Hong, 1996; Dorato *et al.*, 1997)).

The first algorithm which is practically useful has been given in (Collins, 1975), with improvements described in (Arnon *et al.*, 1988; Collins and Hong, 1991; McCallum, 1988). Other algorithmic approaches are described in (Renegar, 1992; Weispfenning, 1998; Dolzmann and Sturm, 1997; González-Vega, 1998).

Many applications, especially in mechanical engineering and in numerical analysis (see (Liska and Steinberg, 1993; Jirstrand, 1997)) lead to QE problems with trigonometric functions involved. The general first order theory of the real numbers with sine and cosine is undecidable: since the zeroes of the sine function are precisely the integral multiples of π, one can find diophantine arithmetic within it, which is undecidable. Fortunately, in any reasonable applications it is possible to reduce the trigonometric problem to a strictly algebraic problem (see also (Gutierrez and Recio, 1998; Hong and Schicho, 1998)). Unfortunately, a straightforward reduction is only possible at the cost of introducing additional variables, which has severe consequences for the computational effort.

Here, we propose another approach: adapting Collins's CAD method to the trigonometric case. The method consists in two phases, *the decomposition phase* and *the evaluation phase.* First, the definition space of the polynomial/trigonometric functions is decomposed cylindrically into cells. On each cell the functions have constant sign. Then the cells on which the initial formula is true are collected. Their definition is reported as an equivalent quantifier-free formula.

During the decomposition process, which is carried out recursively, at each step a variable is dropped by means of a special *projection* operator. Once a decomposition of a lower dimensional space is computed, its cells are *lifted* to a decomposition of the total space.

We try to use in this algorithm modules from Collins's CAD. For this purpose, at each step of decomposition, if the main variable – to be eliminated – is an angle variable (i.e., it occurs as argument to sines and cosines), we transform the set of polynomials by means of the tangent of the half angle: Each sine/cosine is replaced by its formula in terms of the tangent of half of the angle. The numerator of the resulting expression is collected and new variables are substituted for the tangents. The polynomials obtained after this transformation are algebraic with respect to the main variable, so we can apply the normal decomposition procedure.

The decomposition computed using these polynomials gives an incomplete decomposition with respect to the initial variables, because, when using the tangent of the half angle, one must avoid π: here the tangent does not exist. Our method copes with this inconvenient by using a special operator for eliminating a variable, and a special lifting method which leads to a cylindrical decomposition of the whole space.

The decomposition thus generated will not be algebraic anymore, since the cells are computed as sets the boundaries of which are roots of trigonometric polynomials: we call it *cylindrical trigonometric decomposition* The cylindricity property ensures the validity of an algorithm that evaluates the formula in each cell, reporting the collection of cells over which it is true. The final output will be a quantifier-free formula (involving the initial trigonometric polynomials and some other relations) equivalent to the initial one.

The naive approach for solving this problem would be by substituting new variables for each sine and cosine, and adding formulas coresponding to identities of the form $\sin(x)^2+\cos(x)^2 = 1$. This method is much more expensive with respect to the number of operations.

Another approach – not so naive but still insufficient – would be to transform the angles by means of the tangent of their halfs and apply logic to reduce to a finite set of polynomial QE problems, and to use Collins's algorithm (or any other of the above) as a black box. Unfortunately, all CAD-based algorithms assume prenex normal form, and in order to establish this, one needs to introduce additional variables in general. In the worst case, the number of variables may double, hence the situation is as bad as with the naive approach.

A full description of the algorithm has been submitted to *Journal of Symbolic Computation*, under the title **Quantifier Elimination for Trigonometric Polynomials by Cylindrical Trigonometric Decomposition**, authors *Petru Pau and Josef Schicho.*

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