The Parameterization Problem for of Algebraic Surfaces

Josef Schicho*
Research Institute for Symbolic Computation
Johannes Kepler University
A-4040 Linz, Austria
e-mail: `Josef.Schicho@risc.uni-linz.ac.at`
www: `http://www.risc.uni-linz.ac.at/people/jschicho`

June 11, 1999

1 Extended Abstract

We give an overview on the parametrization problem, including recent achievements and open problems.

The parameterization problem is the following. Given is the implicit representation of a surface, in terms of an algebraic equation in three variables. Wanted are three rational functions in two variables, such that the image of the rational map defined by this triple is a two-dimensional subset of the surface. For instance, the the unit sphere with equation $x^2 + y^2 + z^2 - 1 = 0$ has the parameterization

$$(x, y, z) = (\frac{2s}{s^2 + t^2 + 1}, \frac{2t}{s^2 + t^2 + 1}, \frac{s^2 + t^2 - 1}{s^2 + t^2 + 1}),$$

which can be found by stereographic projection.

Over the complex numbers, the well-known criterion of Castelnouvo [7] is very relevant: a complex surface has a rational parametrization iff its arithmetic genus and its second plurigenus both vanish. An algorithm for deciding this criterion and computing a parametrization in case the criterion is fulfilled is contained in [19]. This algorithm is currently being implemented by W. Heiß and the author, and we give a first report on computational experiments.

Over the real numbers, Castelnuovo's criterion is necessary, but it is not known whether it is sufficient for the existence of a parametrization with real

*supported by the Austrian Science Fund in the frame of project P12662-TEC

coefficients. Important subcases have been solved, for instance the case of tubular or conical surfaces (see [18, 15]). This result is relevant for applications in CAD/CAM (see [2, 16]).

Related problems are the parametrization problem for algebraic curves (studied in [26, 1, 20, 17, 22, 14, 13, 21, 23]) and the implicitization problem studied in [6, 9, 11, 8]. Another related problem is the problem of resolution of singularities (see [25, 10, 12, 24, 3, 5, 4]), which is required as a subtask for surface parametrization.

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