Symbolic Computation of Formal Solutions for 2 and 3 Dimensional Dynamical Systems

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Abstract

First we will consider 2-dimensional autonomous dynamical systems

$$\dot{X} = F(X),\ X = \begin{pmatrix} x \\ y \end{pmatrix},\ F = \begin{pmatrix} f \\ g \end{pmatrix},\ f, g \in \mathbb{C}[x, y] \tag{1}$$

and present the implementation of an algorithm proposed by A. Bruno that can be used to find parametrized formal solutions $(x(t), y(t))$ in the neighbourhood of the origin for these systems if the origin is a simple point or an isolated singularity. Then we will give some examples how this algorithm can be extended for the integration of 3-dimensional dynamical systems.

The algorithm is split into three parts according to the classification of the dynamical systems into 3 cases:

- the origin is a simple point if $F(0) \neq 0$.
- it is an elementary singular point if $F(0) = 0$ and the linear part is not nilpotent.
- it is a nonelementary singular point if $F(0) = 0$ and the linear part is nilpotent.

In the case of a simple point we can apply the flow box theorem to transform the initial system into

$$\dot{\tilde{X}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and use Taylor series to find the transformation $X = H(\tilde{X})$.

If the origin is an elementary singular point we can apply normal form constructions to reduce the initial system to an integrable system. We compute the Poincaré-Dulac normal form if the eigenvalues λ_1 and λ_2 of the linear part are in $\mathbb{R}$ or if we have complex coefficients in F. If we have only rational coefficients in F and $\lambda_1, \lambda_2 \in \mathbb{C}$ we calculate the rational normal form which is linear, if λ_1 and λ_2 are purely imaginary we use the rational normal form and polar coordinates.

In the case of a nonelementary singular point the Newton polygon is used to divide the neighbourhood of the origin into sectors. Now we calculate formal solutions valid on these sectors. For all sectors corresponding to vertices of the Newton polygon we apply a new time transformation and a power transformation to transform the initial system to an system that has an elementary singular point in the origin. On the sectors corresponding to edges of the Newton polygon we use directional blowing up to desingularize the singularity. This

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leads to a finite number of systems that can be integrated using the methods described above.

The program has been implemented in Maple. It can be used and tested on-line at `http://www-lmc.imag.fr/lmc-cf/Gerard.Eichenmueller/Software/`. We will treat some examples to illustrate each of the possible cases and to show how the algorithm works.

This algorithm can be applied to integrate 3-dimensional dynamical system

$$\dot{X} = F(X),\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ F = \begin{pmatrix} f \\ g \\ h \end{pmatrix},\ f, g, h \in \mathbb{C}[x, y, z]$$

if the origin is an elementary singular point and the Poincaré-Dulac normal form

$$\left\{ \begin{array}{lll} \frac{dx}{dt} = & \lambda_1 x + \sigma_1 y & +x \sum_{\langle Q,\Lambda\rangle=0} a_Q X^Q \\ \frac{dy}{dt} = & \lambda_2 y + \sigma_2 z & +y \sum_{\langle Q,\Lambda\rangle=0} b_Q X^Q \\ \frac{dz}{dt} = & \lambda_3 z & +z \sum_{\langle Q,\Lambda\rangle=0} c_Q X^Q \end{array} \right.$$

has its support on a plane. That means that the resonance equation

$$\langle Q, \Lambda\rangle = 0$$

has 2 linearly independent solutions in $\mathbb{N}^3$. Then the 3-dimensional system can be reduced to a 2-dimensional system using a change of coordinates described by a power transformation. The resulting sytem has a nonlementary singularity in the origin and we can find a division of the neighbourhood into sectors and formal solution on those sectors. The two-dimensional solutions can be retransformed into (x, y, z) coordinates. But the approbiate power transformation that reduces the system does not always exist. We will show when it is possible to find it and how to construct it if it exists.

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