Computation of Normal Forms of Hamiltonian Systems in the Presence of Poisson Commuting integrals

(Liouville-Integrability and Birkhoff Normal Forms)

J. Mikram & F. Zinoun
Département de Mathématiques et Informatique
Faculté des Sciences de Rabat, Morocco

Consider a Hamiltonian system with n degrees of freedom having the origin as an equilibrium point, and assume that the Hamiltonian function $H(z)$ is of class C^∞ in a neighborhood of the origin. By setting $H(0) = 0$, the Hamiltonian function can be written as

$$H = \frac{1}{2} \langle Sz, z \rangle + O\left(|z|^3\right)$$

and the corresponding Hamiltonian system is

$$\frac{dz}{dt} = JH_z = JS_z + O\left(|z|^2\right), \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1}$$

where S is the Hessian matrix of H at the origin, $\langle ., . \rangle$ denotes the Euclidean inner product of $\mathbf{R}^{2n}$, I is the identity matrix of degree n and $O\left(|z|^d\right)$ is a function whose Taylor expansion at the origin begins with terms of order $\geq d$. If the origin is an elliptic equilibrium point, we can find a (real) linear canonical transformation so that the Hamiltonian function is of the form

$$H = \frac{1}{2} \sum_{j=1}^{n} \lambda_j \left(x_j^2 + y_j^2\right) + O\left(|z|^3\right), \quad \lambda_j \in \mathbf{R}$$

A preliminary reduction to normal form consists in carrying out a canonical change in variables allowing to pass from the Hamiltonian H to a simpler (normal form) $\widetilde{H}$, satisfying

$$adH_2\left(\widetilde{H}\right) = 0$$

adH_2 being the linear operator $\{H_2, -\}$ where $H_2 = \frac{1}{2} \sum\limits_{j=1}^{n} \lambda_j \left(x_j^2 + y_j^2\right)$ and the brackets denote the canonical Poisson bracket

$$\{F, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j} \frac{\partial G}{\partial x_j} \right)$$

The aim of this talk is to show how one can take profitably the existence of first integrals (i.e., quantities F that are preserved by the flow of the Hamiltonian system, or equivalently $\{H,F\}=0$) to produce a simplified normal form.
The motivation for considering the first integrals of the Hamiltonian (1) in view of a further reduction of the normal form comes from the following fact:
Let F be a first integral of (1), that is

$$\{H,F\}=0$$

Suppose we have taken H into a normal form $\widetilde{H}$ via a canonical coordinate transformation T. Then, if we consider the transformation of F under T, writing $F\circ T\equiv\widetilde{F}$ (not necessarily in normal form!), we have

$$\left\{\widetilde{H},\widetilde{F}\right\}=0$$

because T is canonical and then it preserves the Poisson bracket. Therefore, the presence of a first integral for (1) will pose some restriction to the normal form $\widetilde{H}$. In fact, while in general $\widetilde{H}$ only satisfies (with obvious notation)

$$\widetilde{H}\in Ker\left(adH_2\right)\quad ,$$

in the presence of the first integral we will also have

$$\widetilde{H}\in Ker\left(ad\widetilde{F}\right)$$

which can lead to a simplification of the normal form.

This obvious, but important, fact prompts to examine in a more general context for general Hamiltonian structures the role of first integrals in further reducing the normal form unfolding of Hamiltonian systems. The result is that, via a normal form approach, we can reduce simultaneously and in an algorithmic way the Hamiltonian together with its Poisson commuting integrals into a joint normal form which is given a simple characterization.

A prominent class of Hamiltonian systems for illustrating the joint normal form approach are Liouville-integrable systems, i.e., roughly, n degrees of freedom Hamiltonians with n functionally independent and Poisson commuting integrals. We have shown that a number of interesting results on computation of Birkhoff normal forms for Liouville-integrable Hamiltonian systems can be seen as a direct consequence of the joint normal form approach.

As a Computer Algebra Application, we choose an integrable case of the generalized Hénon-Heiles system. This provides an interesting example to show the importance of the joint normal form algorithm in taking to Birkhoff normal form (and then solving) Liouville-integrable Hamiltonian systems.

This work is an extension (and application to physical problems) of a previous work communicated in IMACS-ACA'98. A part of the overall algorithm has been implemented in the computer algebra system MAPLE V.

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