Analytical Approach for Simplifying Dynamical Systems of Polynomial Type

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Many dynamical systems are modelled by a system of ordinary differential equations which is formed by the sum of a linear part plus a small perturbation. Moreover, the perturbation is a finite sum of nonlinear vector fields which begins with quadratic polynomials. In other words, we study systems of the type

$$\frac{d\,\mathbf{x}(t)}{d\,t} \;=\; A\,\mathbf{x}(t) + \sum_{n=1}^{L} \frac{\varepsilon^n}{n!} \mathbf{F}_n(\mathbf{x}(t)), \tag{1}$$

where A is an $m \times m$–matrix with constant coefficients with physical dimensions [time^{-1}], ε stands for a dimensionless small parameter and the vector field $\mathbf{F}_n$ has m components and each component is a polynomial of degree $n+1$.

Analytical methods to deal with systems of the type (**??**) are based on the fact that these differential equations can be understood as perturbations of the principal part, $A\,\mathbf{x}(t)$. In such circumstances, one can think of reducing the problem to a simpler one by means of adequate changes of variables.

These equations appear in many situations of Dynamical Systems Theory, such as stability and bifurcation analysis of equilibrium points and periodic orbits or singularity theories. Besides, applications are encountered in several fields such as Engineering (van der Pol–like and Duffing–like equations, see [1]) or Astrodynamics (potential functions modelling elliptical galaxies, see [7]).

The first step in the analysis of Equation (**??**) was given by Poincaré in [6]. He considered the problem of reducing it to a system of linear equations

$$\frac{d\,\mathbf{y}(t)}{d\,t} \;=\; A\,\mathbf{y}(t),$$

by defining the formal change of variables $\mathbf{y}(t) = \varphi(\mathbf{x}(t); \varepsilon) = \mathbf{x}(t) + \ldots$. Concretely, he found a solution in the case where A were diagonalizable and its eigenvalues $\lambda_1, \ldots, \lambda_m$ satisfied the resonant condition, that is, $\lambda_j \neq \sum k_i\,\lambda_i$ for $j = 1, \ldots, m$ and all integer vectors $k = (k_1, \ldots, k_m)$ with $k_i \geq 0$ and $|k| = k_1 + \ldots + k_m \geq 2$. He also proved that, if in addition to the above, the eigenvalues of A, considered in the complex plane, lie strictly to one side of a line through the origin of the plane, then the formal series which gives the change of variables converges (for a proof see reference [7]).

When these conditions for the eigenvalues are not satisfied, one tries to find the normal form of (??), that is, to reduce this system to an "equivalent" one

$$\frac{d\,\mathbf{y}(t)}{d\,t} \;=\; A\,\mathbf{y}(t) + \sum_{n=1}^{M} \frac{\varepsilon^n}{n!}\mathbf{G}_n(\mathbf{y}(t)) + \mathcal{O}(\varepsilon^{M+1}), \tag{2}$$

where $M \geq L$ and $\mathbf{G}_n$ are "simpler" than $\mathbf{F}_n$ for $n = 1, \ldots, L$.

Meyer [3] has generalized Poincaré's results by giving the Normal Form Theorem, based on Lie transformations. See reference [2] for the method applied to Hamiltonian systems and reference [4] for the generalization to differential equations and tensor fields. We summarize THE theorem briefly. Let $A = A_S + A_N$ be the Jordan decomposition of a square matrix into its semisimple and nilpotent parts, respectively. Let A^t be the transpose matrix of A (which is, indeed, the adjoint matrix for the standard inner product in $\mathbf{R}^m$). Denote by $[\,\cdot\,,\,\cdot\,]$ the Lie bracket of two vector fields, that is, $[\,\mathbf{g}_1\,,\,\mathbf{g}_2\,] = (\partial\mathbf{g}_1/\partial\mathbf{x})\,\mathbf{g}_2 - (\partial\mathbf{g}_2/\partial\mathbf{x})\,\mathbf{g}_1$ and by $\mathcal{L}_A$ the linear operator associated to A, such that, for any vector field $\mathbf{g}$ defined in a domain of $\mathbf{R}^m$, $\mathcal{L}_A(\mathbf{g}) = [\,\mathbf{g}\,,\,A\,\mathbf{x}\,]$. The theorem states that there exists a formal transformation $\mathbf{y}(t) = \varphi(\mathbf{x}(t);\varepsilon) = \mathbf{x}(t) + \ldots$ which transforms (??) into (??), where each term $\mathbf{G}_n$, $n = 1, \ldots, M$ belongs to the kernel of the operator $\mathcal{L}_{A^t}$, that is,

$$[\,\mathbf{G}_n\,,\,A^t\,\mathbf{x}\,] = 0, \quad n = 1, \ldots, M. \tag{3}$$

Note that the transformation is analytic since we truncate the change of variables at an order M. Moreover, by virtue of (??), it is not difficult to see that, $[\,A\,\mathbf{x}\,,\,A_S\,\mathbf{x}\,] = 0$ and $[\,\mathbf{G}_n\,,\,A_S\,\mathbf{x}\,] = 0$, $n = 1, \ldots, M$, since $\ker\mathcal{L}_{A^t} \subset \ker\mathcal{L}_{A_S}$.

However, if $A_S = 0$, the consequence of applying the Normal Form Theorem is not so drastic. Then, the strategy to reduce the system must be different. Moreover, even if $A_S \neq 0$, the above result does not permit to further simplify the initial system. Therefore, one can think of generalizing the calculation of normal forms by Lie transformations to simplify systems of the type (??) even more.

Let T be a square $m \times m$–matrix with Jordan decomposition $T = T_S + T_N$, that is, T_S semisimple and T_N nilpotent. In each step of the process of a Lie transformation ($n = 1, \ldots, M$) one has to solve the homology equation

$$\mathcal{L}_A(\mathbf{W}_n) + \mathbf{G}_n \;=\; \widetilde{\mathbf{F}}_n,$$

where $\mathbf{W}_n$ is related to the n–th order of the generating function whereas $\widetilde{\mathbf{F}}_n$ denotes the vectors computed in the previous steps. Then, we try to find $\mathbf{G}_n$ such that $[\,\mathbf{G}_n\,,\,T\,\mathbf{x}\,] = 0$. For that, we split $\widetilde{\mathbf{F}}_n = \widetilde{\mathbf{F}}_n^{\#} + \widetilde{\mathbf{F}}_n^{\&}$ wh ere $\widetilde{\mathbf{F}}_n^{\#} \in \ker(\mathcal{L}_T)$, i.e., $\mathcal{L}_T(\widetilde{\mathbf{F}}_n^{\#}) = [\,\widetilde{\mathbf{F}}_n^{\#}\,,\,T\,\mathbf{x}\,] = 0$, identify $\mathbf{G}_n = \widetilde{\mathbf{F}}_n^{\#}$ and $\widetilde{\mathbf{F}}_n^{\&} = \widetilde{\mathbf{F}}_n - \widetilde{\mathbf{F}}_n^{\#}$. T he vector $\mathbf{W}_n$ must be a solution of the system of partial differential equations $\mathcal{L}_A(\mathbf{W}_n) = \widetilde{\mathbf{F}}_n^{\&}$.

If, in addition to the above, T has been chosen so that it commutes with A, then $[\,A\,\mathbf{x}\,,\,T\,\mathbf{x}\,] = 0$ and one reduces the original system. Note that this approach enlarges the Normal Form Theorem and generalizes it since one does not need to take A^t but any other matrix T.

One can prove that if T satisfies: i) it is semisimple, that is, $T_N = 0$; ii) $A\,T = T\,A$ and iii) $\ker(\mathcal{L}_A) \cap \mathrm{im}(\mathcal{L}_T) = \{0\}$, then the decomposition of $\widetilde{\mathbf{F}}_n$ is performed straightforwardly. Moreover the system of partial differential equations is solvable and $\mathbf{W}_n$ are polynomial vector fields, for details see [6]. In this "optimal" case one can reduce the initial system to an equivalent one, applying several Lie transformations. Now, the final reduced system can be studied more easily and the reconstruction of the original equations is performed by making use of the inverse Lie transformation up to a certain order M.

If some condition of the latter paragraph does not hold, one must apply the Lie transformations carefully. Now, the process can terminates before, either because it is not possible to find out more semisimple matrices, T, or, because in any Lie transformation, one can compute $\mathbf{W}_n$ only up to a certain order M so that the generating functions are polynomials.

We illustrate the technique with some examples of differential systems putting special emphasis in the implementation of the method with the algebraic processor MATHEMATICA.

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