CONVODE, other improvements

Alain Moussiaux

Département de Physique, Facultés Universitaires Notre-Dame de la Paix, rue de Bruxelles, 61, B-5000 Namur, Belgium

Richard Mairesse

Centre de Calcul, Facultés Universitaires Notre-Dame de la Paix, rue Grandgagnage, 21, B-5000 Namur, Belgium

Abstract

CONVODE is a REDUCE package that resolves O.D.E. , P.D.E. , systems of O.D.E., P.D.E.. . . The use of CONVODE is very simple. The package is constructed on classical methods for finding solutions and also to keep its user aware of the solving process.

1 Introduction

CONVODE [1] is a program under development. In this paper we will show some interesting results that are obtained by the most recent version of the package. CONVODE is written in the REDUCE language. The purpose of CONVODE is to resolve differential equations. CONVODE tackles with the following kind of problems:

- Ordinary differential equations (O.D.E.) of the first order.
  Linear. Exact. Homogeneous. Bernoulli. Ricatti. Abel. Integrating factors . . . ,
- Ordinary differential equations of higher order
  Linear (with or without constant coefficients). Euler, Frobenius method. Equidimentional in y or in x. Scale invariant. Generalization of Euler. Integrating factors . . .
- Some systems of differential equations.

[1] See [1], [2], [4]. In [3], A. Moussiaux has written a part about CONVODE.

With constant coefficients (of the first order or higher and with and without constraints). Some linear systems with not constant coefficients (two unknowns) ...

- Partial differential equations

  Of the first order or higher (depending of the case). With two or three independent variables. With or without constant coefficients. Method of characteristics. Monge's method (see [7]), ...
- Pfaff equations (total derivatives)

  Test of integrability. Integrating factors ...
- Some systems of partial differential equations with constant coefficients.

We will show in the following sections some examples. We reproduce the process of CONVODE and summarize its *comments in framed text.*

2 Ordinary D.E. two examples

2.1 First example

We want to solve a second order nonlinear D.E. (modified 6.178 equation taken from [9])

$$(x-y)x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x(\frac{\mathrm{d}y}{\mathrm{d}x})^2+(x-y)\frac{\mathrm{d}y}{\mathrm{d}x}-y=0 \tag{1}$$

> The equation is scale invariant so we put : $y = s\,x^p, p = 1$ The resulting E.D. is equidimentional in x.

$$\frac{\mathrm{d}^2s}{\mathrm{d}x^2}s\,x-\frac{\mathrm{d}^2s}{\mathrm{d}x^2}x-(\frac{\mathrm{d}s}{\mathrm{d}x})^2x+\frac{\mathrm{d}s}{\mathrm{d}x}s-3\frac{\mathrm{d}s}{\mathrm{d}x}=0 \tag{2}$$

> This equidimentional equation in x is transformed into an autonomous D.E. via the change $t = \log(x)$

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}s-\frac{\mathrm{d}^2s}{\mathrm{d}t^2}-(\frac{\mathrm{d}s}{\mathrm{d}t})^2-2\frac{\mathrm{d}s}{\mathrm{d}t}=0 \tag{3}$$

The autonomous second order D.E. is transformed into a first order D.E. for u with s as independent variable by: $\mathrm{d}s/\mathrm{d}t = u$

$$\frac{\mathrm{d}u}{\mathrm{d}s} u\, s - \frac{\mathrm{d}u}{\mathrm{d}s} u - u^2 - 2u = 0 \tag{4}$$

We solve the first order D.E. for u. In this case the equation is a linear first order D.E.

The solution for u in terms of s is:

$$u = \mathrm{C}_1 s - \mathrm{C}_1 - 2s \tag{5}$$

We use $u = \mathrm{d}s/\mathrm{d}t$ to construct the the first order differential equation for s and t and we solve this first order differential equation. In this example the equation is linear.

$$-\mathrm{C}_1 s + \mathrm{C}_1 + \frac{\mathrm{d}s}{\mathrm{d}t} + 2s = 0 \tag{6}$$

CONVODE restarts in order to solve the equation for s in terms of t the solution is:

$$s = \frac{\mathrm{e}^{\mathrm{C}t}\,\mathrm{C}_1\mathrm{C} - 2\,\mathrm{e}^{\mathrm{C}t}\,\mathrm{C}_1 + \mathrm{e}^{2t}\,\mathrm{C}}{\mathrm{e}^{2t}\,\mathrm{C} - 2\,\mathrm{e}^{2t}} \tag{7}$$

We use the definitions of s and t to go back to the initial variables x and y and write the final solution.

$$y = \frac{x^{\mathrm{C}}\mathrm{C}_1\mathrm{C} - 2x^{\mathrm{C}}\mathrm{C}_1 + \mathrm{C}x^2}{\mathrm{C}x - 2x} \tag{8}$$

This answer is always in the identifier `reponse`. It can be used to check the solutions.

2.2 Second example

Let us assume that we want to solve (6.178 equation in [9]) :

$$(x+y)x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (x-y)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \tag{9}$$

This equation is very similar to equation 1 but CONVODE follows an other way.

The equation is exact and may be transformed into a first order D.E. for y involving a function $h(x, y)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}xy + h(x, y) = \mathrm{C}_1 \tag{10}$$

And in the same time a P.D.E. for h is provided (see [5]).

$$-\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}p - px - 2py - y = 0 \tag{11}$$

A solution $h(x, y)$ independent of p is computed.

$$h(x, y) = -xy - y^2 \tag{12}$$

Substitution in 10 gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}xy - \mathrm{C}_1 - xy - y^2 = 0 \tag{13}$$

To solve the last equation, CONVODE calls itself and chooses the first method that runs.

An integrating factor is found. $$\mu = \frac{1}{x^3} \tag{14}$$

This integrating factor allows integration and the final answer of CONVODE is:

$$\boxed{\frac{\mathrm{C}_1 - 2\mathrm{C}_2x^2 + 2xy + y^2}{2x^2} = 0} \tag{15}$$

The solution is implicit but the solve of REDUCE may be used to write the explicit solution. This is done by hand because the time for this task depends of the implicit form of the solution.

$$y = \pm\sqrt{-\mathrm{C}_1 + 2\mathrm{C}_2x^2 + x^2} - x \tag{16}$$

3 Parametric solution

Let us try to solve with CONVODE 1.534 equation in [9]. This equation is:

$$4x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 - 6y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 3y - x = 0 \tag{17}$$

It is a first order D.E. but not of the first degree in $\mathrm{d}y/\mathrm{d}x$.

The usual way of working of CONVODE will be to solve with respect to $\mathrm{d}y/\mathrm{d}x$. In this example, with the REDUCE switch `fullroots` set to `on`, it is possible but the three resulting equations are very complicated.

A solution under parametric form is in this case more easy to write (p30 in [9] for example).

If the CONVODE switch `firstordersolve` is set to `parametrique`, when a solution is available by CONVODE under parametric form all the computations are stopped after this step.

> The equation is perhaps of Lagrange-Clairaut type? Your equation is not of Lagrange-clairaut type.

Here CONVODE constructs a first order D.E. for y in terms of an independent variable λ.

> It is not so simple to find a parametric solution!

$$G(\lambda, y) = \frac{3y\left(2\lambda^2 - 1\right)}{4\lambda^3 - 1} \tag{18}$$

> Let me construct and solve:
>
> $$\frac{\mathrm{d}y}{\mathrm{d}\lambda} = \frac{\lambda\frac{\mathrm{d}G}{\mathrm{d}\lambda}}{1 - \lambda\frac{\mathrm{d}G}{\mathrm{d}y}} \tag{19}$$

This equation will give y in term of λ CONVODE restarts and writes:

> The solution is:
>
> $$y = 4\mathrm{C}_1\lambda^3 - \mathrm{C}_1 \tag{20}$$

The solution for $x(\lambda)$ is computed from :

$$x(\lambda) = G(y(\lambda), \lambda) \tag{21}$$

and the final solution is:

$$\boxed{x = 6\mathrm{C}_1\lambda^2 - 3\mathrm{C}_1} \tag{22}$$

$$\boxed{y = 4\mathrm{C}_1\lambda^3 - \mathrm{C}_1} \tag{23}$$

If you want to check, use the resultant to eliminate λ between x and y, solve with respect to y and substitute in the given equation.

4 An example of P.D.E.

We want to solve the following P.D.E. with CONVODE (page 221 of [6] with $a = 8$) :

$$(xy + 64)\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = 8\left(x^2 + y^2\right)u^2 \tag{24}$$

Due to the u^2 in the right member the equation in a nonlinear partial differential equation.

The general form of the equation is:

$$f(x, y)\frac{\partial u}{\partial x} + g(x, y)\frac{\partial u}{\partial x} = h(x, y, u) \tag{25}$$

where $h(x, y, u)$ is a nonlinear function.

The package CONVODE tries to solve the equation without the nonlinear term. In this case it reduces to:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0 \tag{26}$$

A lot of test are done by CONVODE and the solution is written in terms of an arbitrary function $arbf()$.

$$\boxed{u = arbf(xy)} \tag{27}$$

This is of course the general solution of 26.

In order to compute the solution of the given equation CONVODE performs the ad hoc change of variable and writes:

$$\boxed{\eta x \frac{\mathrm{d}u}{\mathrm{d}x} + 64x \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{8\eta^2 u^2 + 8u^2 x^4}{x^2}} \tag{28}$$

This is an ordinary D.E. for u in terms of x, and η must be viewed as a parameter.

The equation is separable.

The solution for u in terms of x and η is then:

$$\frac{-\mathrm{C}_1 u x^2 - 4\eta^2 u + \eta x^2 + 4ux^4 + 64x^2}{ux^2} = 0 \tag{29}$$

Now substitution of η by xy (which is the argument of $arbf()$) and replacing C_1 by an arbitrary function $arbfonc()$ of $\eta = xy$ gives the general solution of the problem.

$$\boxed{u = \frac{xy + 64}{arbfonc\,(xy) - 4x^2 + 4y^2}} \tag{30}$$

We have simply to choose an arbitrary function, to compute u with this choice and to substitute in the given equation to check the answer.

5 Pfaff equation

One of the possibilities of CONVODE is also to solve total differential equations (see [7] and [8]).

In the following example we have a problem involving four variables x, y, z, u.

We want to solve:

$$\begin{multline}(e^u ux^2z^2du + e^x ux^3yzdx + e^x ux^2yzdx + e^x ux^3zdy - \\ e^x ux^3ydz + 2\log(u)ux^3z^2dx + 8u^8xyz^2du + x^4z^2du - \\ u^9yz^2dx + u^9xz^2dy)/\left(ux^2z^2\right) = 0\end{multline} \tag{31}$$

The comments of CONVODE are the following:

You have a total differential equation with 4 independent variables.

Is the problem integrable? Yes, the problem is integrable.

Is the equation exact? Yes the equation is exact and the solution is:

$$\frac{e^u x z + e^x x^2 y + \log(u)\, x^3 z + u^8 y z}{x z} = \mathrm{C} \tag{32}$$

Of course it is not always so simple as shown by the next example.

$$x dx + y dy + x^2 z dz + y^2 z dz + z^3 dz + z dz = 0 \tag{33}$$

Here we have only three independent variables x, y, z

The comments of CONVODE are clear:

You have a total differential equation with 3 independent variables.

The problem is integrable, but the equation is not exact.

An integrating factor is:

$$\mu = e^{z^2} \tag{34}$$

Now the procedure restarts after multiplication by the integrating factor and the solution is:

$$e^{z^2} x^2 + e^{z^2} y^2 + e^{z^2} z^2 = \mathrm{C} \tag{35}$$

CONVODE may also solve P.D.E. involving three independent variables x, y, z or linear P.D.E. of the second order with two independent variables. In this later case, a classification into parabolic, elliptic and hyperbolic equations is done by CONVODE and reduction to the canonical form is also one step of the computations performed by CONVODE.

6 Use of CONVODE by E-Mail

CONVODE may be used via E-mail. All informations are available at the following adress:

`www.physique.fundp.ac.be/physdpt/administration/convode.html`

The use of this facility is very simple.

The following equation from [10]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(\mathrm{e}^{2x} + x\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{1}{2}\left(2\,\mathrm{e}^{2x} + 1\right) - \frac{1}{4}\left(\mathrm{e}^{2x} + x\right)^2\right) y = \frac{\log x}{x+2} \qquad (36)$$

You write a data file of the form:

```
nmax:=2;
smax:=3;
argstop:=1;
lparti:={};
depend y,x;
l1:={(x+2)*df(y,x,2)=(e**(2*x)+x)*(x+2)*df(y,x)+
((1/2)*(2*e**(2*x)+1)-(1/4)*(e**(2*x)+x)**2)*(x+2)*y+log(x)};
l2:={y};
l3:={x};
l4:={};
l5:={};
convode(l1,l2,l3,l4,l5);
```

and you send this file to: `convode@riemann.physmath.fundp.ac.be`. After a few minutes you will receive a mail from the workstation `riemann` with, I hope, the solution with the comments of CONVODE.

The five arguments of CONVODE (`l1`, `l2`, `l3`, `l4`, `l5`) must be REDUCE lists. The equation to solve is in `l1`, the unknown(s) in `l2`, the independent variable(s) in `l3`. The initial conditions must be introduced in the list `l4` and if `l5`:={english}; the comments are in principle written in english.

Before `l1`, the conveniant dependencies must be introduced using the REDUCE `depend` instruction(s).

The meaning of the parameters is the following:

`nmax` is a positif integer: CONVODE searchs for polynomial solution of maximum degree `nmax`.

`smax` is also a positif integer which fixes the number of terms computed when the solution provided by CONVODE is a series.

`lparti` is used when for second order linear D.E., you already know one of the solutions (`lparti`:={y=x};) in this case CONVODE computes the second solution.

`argstop` is an integer lower than `nmax`. If for example the first solution of a linear D.E. of the second order is a polynomial of degree 6, the second solution may be very complicated and perhaps not interesting, if `argstop` is lower than 6 the second solution is not computed by CONVODE. This was introduced for people interested only by polynomial solutions.

7 Conclusion

The purpose of CONVODE is to provide a package for solving O.D.E., P.D.E., systems of D.E., Pfaff equations and so on. In the case of P.D.E. we are interested by the general solution of the problem involving arbitrary functions.

The package is written in such a way that the user may follow the process from the input to the final result.

In this paper we have try to show on some examples the step by step approach of CONVODE. It is clear that CONVODE is a recursive package. It is also clear that the possibilities of CONVODE are limited and that any user may submit an equation intractable for CONVODE, nevertheless many kinds of classical problems may be solved by our package.

By design, CONVODE is not a "black box" providing a solution without comments. In the opposite, we know the way followed by CONVODE to provide a solution. We do not claim that this way is always the best but we want also preserve the "pedagogical" look of our package for use by students. When it does'nt find a solution, the work done can be fruitful.

References

[1] A. Moussiaux. *CONVODE:un programme REDUCE pour la résolution des équations différentielles.* (Didier Hatier, Bruxelles, 1994).

[2] A. Moussiaux. *CONVODE: a REDUCE package for solving differential equations.* (Journal of Computatinal and Applied Mathematics 48(1993), Elsevier Sciences Publisher.Amsterdam The Netherland, 1993).

[3] A.D. Polyanin, A.V. Vyazmin, A.I. Zhurov and D.A. Kazenin. *Handbook of Exact Solutions of Heat and Mass Transfer Equations [in Russian].* (Faktorial, Moscow, 1998).

[4] A. Moussiaux, R. Mairesse. *CONVODE: new possibilities.* (XI Russian Colloquium on Mathematical Modeling, University of Samara, 1993).

[5] Staff of Research and Education Association. *Advanced Methods for solving differential equations.* (R.E.A, New-York, 1982).

[6] V.F. Zaitsev, A.D. Polyanin. *Handbook of Partial Differential Equations: Exact solutions [in Russian].* (Mezhdunarodnaya Programma Obrazovania , Moscow, 1996).

[7] D. Zwillinger. *Handbook of differential equations.* (Academic Press, Inc., Boston, New-York,..., 1992).

[8] E.Goursat. *Differential equations, Volume II, Part II.* (Dover Publications, inc., New York, 1959).

[9] E. Kamke. *Differentialgleichungen Losungsmethoden und Losungen.* (Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1956).

[10] E.S. Cheb-Terrab, A.D. Roche *Integrating Factors and ODE Patterns* (Preprint IF-UERJ-2/97, November 1997).