On the homology of semidirect product of groups.

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Abstract

We are concerned here with the establishment of an algorithm for computing the homology of semidirect products of both simplicial and discrete groups. We also analyze some complexity aspects on this algorithm. Finally, we compare our method with the one in [10] for obtaining the homology of dihedrical groups. A practical implementation of the algorithm in *Mathematica 3.0* is expected to be provided.

1 Extended Abstract.

We describe here an algorithm for computing the homology of certain semidirect products $A \times_\chi G$ of both discrete and simplicial groups.

In [8], Lambe and Stasheff described a method for computing the homology of central extension of groups. This method has been applied to finitely generated torsion-free nilpotent groups [8, 4, 5] finite p-groups [6, 7, 1]. Using a related perturbation method, Huebschmann in [2] and [3] attacks, respectively, the problem of the homology computation of finitely generated two-step nilpotent groups and metacyclic groups.

More precisely, in [8], a simplicial model $\bar{W}(G)$ of the classifying space of a central extension of groups G is "factorized" and given in the form of twisted cartesian product. The application of the twisted Eilenberg-Zilber theorem, some annihilation properties of the corresponding twisting cochain and the existence of a "homological model" for the group allow the construction of a small free resolution for finitely generated torsion-free nilpotent groups.

We follow here a similar procedure to that in [8]. In fact, [8] has provided much inspiration to this work.

Let A and G be two groups and $\chi : G \times A \longrightarrow A$ be an action from G to A. The *semidirect product* of A and G, denoted by $A \times_\chi G$, is the group where the underlying set is just $A \times G$ and the operation is given by the following rule:

$$(a, g) \cdot (a', g') = (a + \chi(g, a'), g + g').$$

For the remainder of the paper, we write ga' to refer to $\chi(g, a')$.

The map χ is called a *group action* if the following condition holds:

$$g(a + a') = ga + ga' \;\; \forall g \in G, \;\; \forall a, a' \in A.$$

In an obvious way, the notion of semidirect product can be extended for simplicial groups.

Let consider the twisted cartesian product (briefly, TCP) $\bar{W}(A) \times_\tau \bar{W}(G)$ with structural group G via the action

$$\begin{array}{rcl} \kappa \, : \, G \times \bar{W}(A) & \longrightarrow & \bar{W}(A) \\ (g, [a_{n-1}, \ldots, a_0]) & \longrightarrow & [g \cdot a_{n-1}, \ldots, g \cdot a_0]; \end{array}$$

and twisting function $\tau_n : \bar{W}_n(G) \longrightarrow G_{n-1}$,

$$\tau_n[g_{n-1}, \ldots, g_0] = g_{n-1}.$$

Theorem 1.1. *In the conditions above, there is an explicit simplicial isomorphism*

$$f : \bar{W}(A \times_\chi G) \longrightarrow \bar{W}(A) \times_\tau \bar{W}(G)$$

Though the TCP $\bar{W}(A) \times_\tau \bar{W}(G)$ defined by a semidirect product of simplicial groups $A \times_\chi G$ is not principal, proof of Theorem 3.5 in [8] may be translated to our settle, so that an homological model for certain semidirect product of groups is so constructed.

Theorem 1.2. *Let $A \times_\chi G$ be a semidirect product of simplicial groups, so that the twisting function t on $\bar{B}(A) \otimes_t \bar{B}(G)$ induced by τ becomes null acting upon elements of degree 1. Then, there is an homological model for $A \times \chi G$, which is related to the existence of homological models for the factors A and G.*

The hypothesis on the above theorem are not satisfied in the case of simplicial groups associated to discrete groups. In spite of this fact, we may take advantage of some properties of t, so that the application of some basic homological perturbation machine leads to an extended version of the previous theorem to the case of discrete groups.

Theorem 1.3. *Let A be a discrete group having an homological model, G be a finite abelian group and $\chi : G \times A \longrightarrow A$ be a group action. Then, $A \times_\chi G$ has an homological model.*

References

[1] T. Ekedahl, J. Grabmeier and L. Lambe, Algorithms for algebraic computations with applications to the cohomology of finite p-groups, *Preprint of Department of Math. and Center for Innovative Computation, University of Wales* 1997.

[2] J. Huebschmann, Cohomology of nilpotent groups of class 2, *J. Alg.* **126** (1989) 400–450.

[3] J. Huebschmann, Cohomology of metacyclic groups, *Transactions of the American Mathematical Society* **328** n. 1 (1991) 1–72.

[4] L. A. Lambe, Algorithms for the homology of nilpotent groups, *Conf. on applications of computers to Geom. and Top., Lecture Notes in Pure and Applied Math.* **114**, Marcel Dekker Inc., N.Y., 1989.

[5] L. A. Lambe, Resolutions via homological perturbation, *J. Symbolic Comp.* **12** (1991) 71–87.

[6] L. A. Lambe, Homological perturbation theory, Hochschild homology and formal groups, *Proc. Conference on Deformation Theory and Quantization with Applications to Physics, Amherst, MA, June 1990, Cont. Math.* **134** A.M.S (1992) 183–218.

[7] L. Lambe, An algorithm for calculating cocycles, *Preprint of Department of Mathematics and Center for Innovative Computation, University of Wales* , 1997.

[8] L. Lambe and J. Stasheff, Applications of perturbation theory to iterated fibrations, *Manuscripta Math.* **58** (1987) 367–376.

[9] J.P. May, Simplicial objects in Algebraic Topology, *Chicago Lectures in Mathematics* , The University of Chicago Press, Chicago and London, 1992.

[10] C.A. Weibel, An introduction to homological algebra, *Cambridge studies in advanced mathematics*, **38**, Cambridge University Press, 1994.