Analyzing the transference of the coalgebra structure on the homology of CDGAs

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Abstract

Our motivation here is to analyze the A_∞-coalgebra structure $(\Delta_2, \Delta_3, \ldots)$ of the small homological model H obtained in [2], for any commutative differential graded algebra (or briefly CDGA) A. More precisely, making use of the facts that H is a CDGA and that the morphism $\Delta_2 : H \to H \otimes H$ satisfies a compatibility condition with regard to the product on H, we design an algorithm testing the coassociativity of Δ_2. Considerations concerning the complexity of this algorithm are given.

1 Extended Abstract

The description of efficient algorithms for homological computation can be considered as a very important question in Homological Algebra, in order to use those processes mainly in the resolution of problems on Algebraic Topology. We are interested in obtaining the A_∞-coalgebra structures of small homological models of Eilenberg-MacLane spaces. These spaces are a kind of prime spaces in homotopy theory and its homology and cohomology are essential in order to compute homotopy invariants of spaces and cohomology operations. In order to progress in this way, a first step is the analysis of the transference of the coalgebra structure in the homology of CDGAs.

Our main technique is the use of Homological Perturbation Theory which is often used to replace given chain complexes by homotopic, smaller and more readily computable chain complexes ([7], [8],[9]). The Basic Perturbation Lemma (BPL) states that given a contraction

$$(f, g, \phi) : (N, d_N) \Rightarrow (M, d_M) \tag{1}$$

of chain complexes (with $fg = 1_M$ and $1_N - gf = d_N\phi + \phi d_N$) and a perturbation δ of d_N (that is, $(d_N + \delta)^2 = 0$), under suitable conditions there exists a perturbation d_δ of d_M such that $H_*(M, d_M + d_\delta) = H_*(N, d_N + \delta)$.

Perturbation results about preservation of structures (DG-algebras, DG-coalgebras, Lie algebras, ...) have been largely considered ([9] and [10]). The technique for obtaining these results is to determine under what conditions the BPL allows the preservation of the data structures.

In the case of algebra contractions, the study of the preservation of the multiplicative structure of the first algebra via the contraction leads to the definition of a special type of contractions: *semi-full algebra contractions.* The semi-fullness is an hereditary property under composition, tensor product and perturbation of contractions (which are new contractions).

Apart from the intrinsic interest of the transference problem, when we deal with DG-algebras or DG-coalgebras, these kind of results allow us to determine and construct essential algebra or coalgebra structures on various chain complexes which have been constructed via perturbation. For example, the homology theory of commutative DGA-algebras is entirely examined in terms of semifull algebra contractions in [2], and "small" p-local homological models of reduced bar constructions of twisted tensor product of Cartan's elementary complexes are obtained in [4], due to the fact that all the contractions appearing there, are of this type. These examples will be our starting point in this paper, being the transference of the coalgebra structure in the previous contractions our main objective.

Let us begin by explaining the algorithm for computing small homological models of CDGAs.

It is commonly known that every commutative DGA A "factors" up to homotopy equivalence into a twisted tensor product of exterior and polynomial algebras endowed with a differential-derivation (see, for example, [12]); in the sense that there exists an homomorphism connecting both structures, which induces an isomorphism in homology. This last TTP is called a *free model* for A.

In fact, the object we start from is the free model of a finite (as an algebra) CDGA A.

The principal goal that arises is to obtain a "chain" of contractions starting at the reduced bar construction $\bar{B}(A)$ and ending up at a commutative DGA-algebra that is free and of finite type as graded module.

Three *almost-full algebra contractions* (i.e., semifull algebra contractions endowed with multiplicative projections) are used in order to find the structure of graded module of a homological model for the DGA-algebra A:

- The contraction defined in [6] from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$, where A and A' are two commutative augmented DGAs; which is briefly denoted by $C_{\bar{B}\otimes}$.

  Though Eilenberg and Mac Lane only set explicit formulae for the projection and inclusion morphisms, an explicit formula of the recursive definition of the homotopy operator given by Eilenberg and Mac Lane in [5] is established in [14] (see also [13] for a proof).

  Given a tensor product $\otimes_{i \in I} A_i$ of augmented CDGA-algebras, a contraction from $\bar{B}(\otimes_{i \in I} A_i)$ to $\otimes_{i \in I} \bar{B}(A_i)$ is easily determined, by applying $C_{\bar{B}\otimes}$ several times in a suitable way. This new contraction is also denoted by $C_{\bar{B}\otimes}$.

- The isomorphism of DGA (hence, a contraction) $C_{\bar{B}E}$ from $\bar{B}(E(u, 2n+1))$ to $\Gamma(\sigma(u), 2n+2)$, also described in [6]. Note that the generator v of the previous divided power algebra is denoted by $\sigma(u)$ where σ is the Cartan's suspension operator.

  Last isomorphism might be considered as a *full algebra contraction* (that is, an almost-full algebra contraction endowed with an algebra homotopy operator), in an obvious way.

- The contraction $C_{\bar{B}P}$ from $\bar{B}(P(y, 2n))$ to $E(\sigma(y), 2n)$ (see [6]). The generator of the exterior algebra is denoted by $\sigma(y)$ since both elements correspond to each other by the projection and inclusion of the contraction.

All the previous contractions have already been described for decades, so we will not explain them further.

Thanks to these three contractions, it is possible to establish, by composition, the following semifull algebra contraction $C_0 = (f, g, \phi)$:

$$\bar{B}(\otimes_{i \in I} A_i) \;\Rightarrow\; \otimes_{i \in I} \bar{B}(A_i) \;\Rightarrow\; \otimes_{i \in I} HBA_i,$$

where HBA_i represents an exterior or a divided polynomial algebra, depending on the fact that A_i is a polynomial or an exterior algebra. Obviously, the product on $\otimes_{i \in I} HBA_i$ is the natural one.

The next step is perturbing C_0, in order to obtain an homological model of the initial TTP $\tilde{\otimes}^{\rho}_{i \in I} A_i$. The morphism ρ adds a perturbation-derivation δ onto the tensor differential of $\bar{B}(\otimes_{i \in I} A_i)$.

Therefore, a new semifull algebra contraction $(f_\delta, g_\delta, \phi_\delta)$ is obtained:

$$\bar{B}(\tilde{\otimes}^{\rho}_{i \in I} A_i) \overset{(C_0)_\delta}{\Rightarrow} (\otimes_{i \in I} HBA_i, d_\delta),$$

where the differential d_δ is determined by the perturbation process. That means that $HBA = \otimes_{i \in I}(HBA_i, d_\delta)$ is a **homological model** of $A = \tilde{\otimes}_{i \in I} A_i$.

We deal with the transference of the coalgebra structure from the reduced bar construction of a CDGA (which has a canonical coalgebra structure) to the homological model described above.

Let us consider the homological model HBA for a CDGA A. That means that we have an algebra contraction:

$$(f, g, \phi) : \bar{B}(A) \overset{C}{\Rightarrow} HBA$$

Let Δ be the natural coproduct on $\bar{B}(A)$:

$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n]$$

This coproduct induces, in a natural way, a morphism on the homological model of A, that is,

$$\Delta_2 = (f \otimes f)\Delta g : HBA \to HBA \otimes HBA \tag{2}$$

We treat the problem of determining if Δ_2 is a real coproduct on the homological model. That means that we analyze whether Δ_2 is coassociative or not.

At first, it would be necessary to evaluate Δ_2 over all the module generators of HBA. Nevertheless, since the projection f is a morphism of CDGAs (see [3]), it is easy to prove that Δ_2 satisfies the following condition:

$$\Delta_2 \circ \mu = (\mu \otimes \mu)(1 \otimes T \otimes 1)(\Delta_2 \otimes \Delta_2),$$

where $\mu : HBA \otimes HBA \to HBA$ is the product of the CDGA HBA and $T : HBA \otimes HBA \to HBA \otimes HBA$ is the morphism that interchanges the factors. In particular, that relation means that for determining the morphism Δ_2, we only need to evaluate it over the generators of HBA as an algebra (a finite number of elements!). Moreover, we attack the problem of the complexity of the associated algorithm to the explicit formula of Δ_2 and we considerably reduce the amount of elementary operations (comparing it to the naive algorithm derived for the initial formulation (2)) that are necessary in order to compute this morphism over an algebra generator.

Finally, we prove that if Δ_2 is coassociative for the algebra generators of HBA, then Δ_2 is coassociative for any element of HBA. This gives us the possibility to design an algorithm testing the coassociativity of this morphism. Several examples "tracing" the previous algorithms in concrete cases are given.

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