THE IDEAL GENERATION CONJECTURE FOR GENERAL RATIONAL PROJECTIVE CURVES

Ferruccio Orecchia

Abstract. We pose a conjecture for the expected number of generators of the ideal of the union C of s general rational irreducible curves in $\mathbb{P}^r$. By using the computer we prove the conjecture for C of low degree d (e.g. if $s=1$ for $d \leq 80$ and if $s \leq 10$ for $d \leq 40$).

1. Introduction.

Let $C = C_1 \cup ... \cup C_s \subset \mathbb{P}^r_k$, $r \geq 3$ be a non-degenerate curve union of s, $s \geq 1$ rational irreducible smooth disjoint curves C_i of degree d_i, $i = 1, ..., s$, over an algebraically closed field k. Then the degree of C is $d = \sum_{i=1}^{s} d_i$.

The curve C has maximal rank if, for every integer $n \geq 1$, the natural restriction map

$$\rho(n) : H^0(O_{\mathbb{P}^r_k}(n)) \to H^0(O_C(n))$$

has maximal rank i.e. $\rho(n)$ is injective or surjective.

Let $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$

A curve C of maximal rank is minimally generated if the natural map

$$\sigma(\alpha) : H^0(I_C(\alpha)) \otimes H^0(O_{\mathbb{P}^r_k}(1)) \to H^0(I_C(\alpha+1))$$

is of maximal rank. Fixed $r, s, d_1, ..., d_s$, if there exists a curve C of maximal rank (respectively of maximal rank and minimally generated), then, for general $C_1, ..., C_s$, C has maximal rank (respectively has maximal rank and is minimally generated) [Corollary 2.6].

The problem of finding the values of $r, s, d_1, ..., d_s$ for which there exists a curve C of maximal rank has been completely solved. In fact, starting from work of Hartshorne and Hirshowitz it has been shown in [4] that if $r \geq 4$ then, for any $s, d_1, ..., d_s$, there exists a curve C maximal rank. Furthermore in [1] it has been shown that, if $r=3$, there exists C of maximal rank, for any $s, d_1, ..., d_s$, except the following five cases: $s=2, d_1=d_2=2$; $s=2, d_1=2, d_2=4$; $s=3, d_1=d_2=d_3=2$; $s=3, d_1=d_2=2, d_3=4$; $s=4, d_1=d_2=d_3=d_4=2$.

The problem of finding the values of $r, s, d_1, ..., d_s$ for which there exists a curve C of maximal rank and minimally generated is, in general, unsolved. In the case of lines, i.e. $d_1 = ... = d_s = 1$, of $\mathbb{P}^3$, it has been shown in [9] that, for any $s \neq 4$, there exist s lines with maximal rank and minimally generated. In the case $s=1$,

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recently in [3] and [10] it has been shown that for any $d \geq 932$ or $d \leq 73$, ($d \neq 5$) there exists an irreducible rational smooth curve of $\mathbb{P}^3$ which has maximal rank and is minimally generated.

In this paper we pose the following conjecture, which we call *ideal generation conjecture* (I.G.C. for short) for rational curves.

For any $r, s, d_1, ..., d_s$, there exists a curve C of maximal rank and minimally generated, except the following s-uples $(d_1, ..., d_s)$.

If $r = 3$: (5) (i.e. $s = 1$ and $d_1 = 5$); $(1, 2)$; $(2, 2)$; $(2, 3)$; $(2, 4)$; $(3, 4)$; $(1, 1, 2)$; $(1, 2, 2)$; $(2, 2, 2)$; $(2, 2, 3)$; $(2, 2, 4)$; $(1, 1, 1, 1)$; $(1, 2, 2, 2)$; $(2, 2, 2, 2)$; $(1, 2, 2, 2, 2)$.

If $r = 4$: (5); $(2, 3)$; $(1, 1, 2)$.

If $r = 5$: $(3, 4)$; $(2, 2, 2)$;

In particular, if $s = 1$, the conjecture is that, for any d and r there exists an irreducible rational smooth curve of degree d in $\mathbb{P}^r$ with maximal rank and minimally generated except when $d = 5$ in $\mathbb{P}^3$ or $\mathbb{P}^4$. This, if proved, would extend to $\mathbb{P}^r$ the result of [3] and [10] in $\mathbb{P}^3$. Moreover, in the case of lines, if the conjecture is proved, we have that for any s and r there exist s lines in $\mathbb{P}^r$ with maximal rank and minimally generated except the case of 4 lines in $\mathbb{P}^3$, thus extending the result of [9] in $\mathbb{P}^3$.

The conjecture has been formulated with the help of the computer. In fact we show how to check the conjecture by computer for fixed $r, s, d_1, ..., d_s$ using an algorithm [Algorithm 3.5] that reduces the computation of the generators of the ideal of the curve C to the computation of the generators for the ideal of a set of points suitably chosen on C. This has allowed us to prove the conjecture on our Personal Computer for $d \leq 80$, if $s = 1$, and for $d \leq 40$, if $s \leq 10$, but enlarging this bound is only a question of having time and/or more powerful computers [Remark 3.10].

Note that the usual systems for computations in Commutative Algebra and Algebraic Geometry like CoCoA [5] seem less suitable for this particular kind of computation [Remark 3.10].

This paper has been inspired by the ideal generation conjecture for a finite set of points in general position, which was posed in [6] and although proved in many cases (see [2] and [8]), is still open in its generality.

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1. Maximal rank.

In all the paper k is a fixed algebraically closed field.

We recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is n-regular if $H^i(\mathcal{F}(n-i)) = 0$ for $i > 0$. If $\mathcal{F}$ is n-regular $\mathcal{F}$ is $n+1$-regular [11, Lecture 14].

Definition 1.1. Let Y be a closed subvariety (also reducible) of $\mathbb{P}^r$ and $\rho(n)$: $H^0(\mathcal{O}_{\mathbb{P}^r_k}(n)) \to H^0(\mathcal{O}_Y(n))$ be the natural restriction map. We say that Y has *maximal rank* if, for every integer $n \geq 0$, $\rho(n)$ has maximal rank as a map of vector spaces i.e. it is injective or surjective.

Let $I(Y) = \bigoplus_{n\geq 0} I(Y)_n \subset k[X_0, ..., X_r] = R$ be the ideal of Y and $A = R/I(Y)$ be the coordinate ring of Y.

Set $H_Y(n) = dim_k A_n = dim_k R_n - dim_k I(Y)_n = \binom{n+r}{r} - dim_k I(Y)_n$ for the Hilbert function of Y and $P_Y(n)$ for the Hilbert polynomial of Y.

In the following we say that Y *is generated in degree* n if the ideal $I(Y)$ of Y can be generated by forms of degree $\leq n$.

Lemma 1.2. *Let Y be a curve and $n \geq 0$ a given integer. Y is n-regular if and only if $\mathcal{O}_Y(n-2)$ is non-special and $\rho(n-1) : H^0(\mathcal{O}_{\mathbb{P}^r_k}(n-1)) \to H^0(\mathcal{O}_Y(n-1))$ is surjective in which case Y is generated in degree n.*

Proof. [7, section 1] □

Proposition 1.3. *Let $C = C_1 \cup ... \cup C_s \subset \mathbb{P}^r_k$, $r \geq 3$ be the union of s rational irreducible smooth disjoint curves C_i of degree d_i, $i = 1, ..., s$. Denote with d the degree $\sum_{i=1}^s d_i$ of C. Then:*

(a) $H_C(n) \leq P_C(n)$ for any n;

(b) if $H_C(m) = P_C(m)$ for some integer m then C is $m+1$-regular, whence the ideal $I(C)$ of C is generated in degree $\leq m+1$ and $H_C(n) = P_C(n)$ for any $n \geq m$;

(c) $P_C(n) = dn + s$;

(d) $H_C(n) \leq Min\{\binom{n+r}{r}, dn+s\}$ and C has maximal rank if and only if $H_C(n) = Min\{\binom{n+r}{r}, dn+s\}$

Proof. (a), (b) Since C is rational the line bundle $\mathcal{O}_C(n)$ is non-special for any n. Then C is n-regular if and only if $\rho(n-1)$ is surjective [Lemma 1.2]. Moreover, for any n, $h^1(\mathcal{O}_C(n)) = 0$ and then $h^0(\mathcal{O}_C(n)) = P(n)$. Hence, for any n, $H(A, n) = dim_k(Im_{\rho(n)}) \leq h^0(\mathcal{O}_C(n)) = P(n)$ (which gives (a)) and $H_C(m) = P_C(m)$, for some integer m, if and only if C is $m+1$-regular. This gives (b) in virtue of Lemma 1.2 and the fact that if C is m-regular, for some integer m, then C is $m+1$-regular.

(c) Since the irreducible curves C_i, $i = 1, ..., s$ are smooth, rational of degree d_i it is well known that $P_{C_i}(n) = d_i n + 1$. Moreover the C_i are disjoint then $P_C(n) = \sum_{i=1}^s P_{C_i}(n) = \sum_{i=1}^s (d_i n + 1) = dn + s$.

(d) The first claim follows from (a) and (c) since $H_C(n) \leq \binom{n+r}{r}$ for any n. The second claim follows by observing that $\rho(n)$ is injective if and only if $H_C(n) = \binom{n+r}{r}$ and that $\rho(n)$ is surjective if and only if $H_C(n) = P_C(n) = dn + s$ by the previous considerations. □

In general the Hilbert function of a rational curve (not smooth) is not bounded by the corresponding Hilbert polynomial as the following example shows.

Example 1.4. *Let $x = u(u^5 - v^5), y = v(u^5 - v^5), z = v^6$ be a parametrization of the rational sextic $C : y^6 - x^5 z + y^5 z = 0$. Then it is easily seen that $P_C(n) = 6n - 9$ while $H_C(3) = 10 > P_C(3) = 9$.*

Proposition 1.5. *Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$ be a non-degenerate curve, such that C_i, are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1 \to \mathbb{P}^r$ given by forms of degree d_i, for any i. Let $d = \sum_{i=1}^s d_i$ and assume that $H_C(m) = dm + s$ for some m. Then C_i are smooth disjoint curves of degree d_i and C is $m+1$-regular.*

Proof. It is enough to prove that the curves C_i are smooth and disjoint since then C is $m+1$-regular by Proposition 1.3.

First we consider the case $i = 1$.

Let C' be the n-uple embedding of C. clearly $C' \subset \mathbb{P}^{nd}$ and $H_C(m) = dm + 1$ is the Hilbert function of the coordinate ring of C' in degree one. Then C' is non degenerate. Hence C' is the rational normal curve of degree nd in $\mathbb{P}^{nd}$. Since C' is smooth also C is smooth and has degree d.

Let now $i > 1$.

If C_i^* is the normalization of C_i then $H_{C_i}(n) \le H_{C_i^*}(n) \le d_i n + 1$, for any n [Proposition 1.3]. Then $md + s = H_C(m) \le \sum_1^s H_{C_i}(m) \le \sum_1^s (d_i m + 1) = dm + s$ and then $H_{C_i}(m) = d_i m + 1$ for any $i = 1, ..., s$. Hence the curves C_i are smooth of degree d_i and C has degree d. Now suppose that C_i and C_j are not disjoint for some i, j. It is easily seen that in this case $H_C(m) < \sum_1^s H_{C_i}(m) = \sum_1^s (d_i m + 1) = dm + s$ and this contradicts the hypothesis that $H_C(m) = dm + s$. □

Theorem 1.6. *Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$ be a non-degenerate curve, such that C_i, are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1 \to \mathbb{P}^r$ given by forms of degree d_i, for any i. Let $d = \sum_{i=1}^s d_i$ and $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$. Then the following conditions are equivalent:*

(a) the curves C_i are smooth of degree d_i and disjoint for any i and C has maximal rank;

(b) $H_C(\alpha - 1) = \binom{\alpha+r-1}{r}$ and $H_C(\alpha) = d\alpha + s$;

and imply that $I(C)$ can be generated by forms of degree α and $\alpha + 1$.

Proof. If $H_C(\alpha - 1) = \binom{\alpha+r-1}{r}$ then $H_C(n) = \binom{n+r}{r}$ for any $n < \alpha$. Hence the claim follows from Propositions 1.3 and 1.5 □

2. Minimal generation.

Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$, $r \ge 3$, be the union of s rational irreducible smooth disjoint curves and denote with $\nu(C)$ the minimal number of generators of the ideal $I(C)$ of C. With I_C we denote the ideal sheaf of C. Assume that C has maximal rank and let $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$; By Theorem 1.6 the ideal $I(C) = \bigoplus_{n \ge 0} I(C)_n$ of C can be generated by forms of degree α and $\alpha + 1$. Then if we set

$$W_{\alpha+1} = X_0 I(C)_\alpha + ... + X_r I(C)_\alpha \subset I(C)_{\alpha+1}$$

we have

$$\nu(I) = dim_k I(C)_\alpha + dim_k I(C)_{\alpha+1} - dim_k W_{\alpha+1}$$

Furthermore:

(1) $dim_k W_{\alpha+1} \le Min\{(r+1) dim_k I(C)_\alpha, dim_k I(C)_{\alpha+1}\}$ and then:

$$\nu(I(C))) \ge dim_k I(C)_\alpha + dim_k I(C)_{\alpha+1} - Min\{(r+1) dim_k I(C)_\alpha, dim_k I(C)_{\alpha+1}\}$$

Definition 2.1. C of maximal rank is minimally generated if:

$$\nu(I(C))) = dim_k I(C)_\alpha + dim_k I(C)_{\alpha+1} - Min\{(r+1) dim_k I(C)_\alpha, dim_k I(C)_{\alpha+1}\}$$

that is

$$dim_k W_{\alpha+1} = Min\{(r+1)dim_k I(C)_\alpha, dim_k I(C)_{\alpha+1}\}$$

that is

$$\sigma(\alpha) : H^0(I_C(\alpha)) \otimes H^0(O_{\mathbb{P}^r_k}(1)) \to H^0(I_C(\alpha+1))$$

is of maximal rank.

We want to show that maximal rank and minimal generation can be checked by computing the generators of a suitably chosen set of points on C.

Lemma 2.2. *Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r_k$ be a non-degenerate curve such that the C_i are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1_k \to \mathbb{P}^r_k$, $\Phi_i([t_1, t_2]) = [f_{i0}(t_1, t_2), ..., f_{in}(t_1, t_2)]$, where $f_{ij} \in k[t_1, t_2]$ are homogeneous polynomials of the same degree d_i. Let $m > 1$ a fixed integer. Consider the set V_i consisting of $h_i = r_i m + 1$ distinct points of C_i, for any i, and let $V = \cup_{i=1}^s V_i$ be the set of all these ($h = \sum_{i=1}^s h_i$) points and $I(V) = \oplus_{n\geq 0} I(V)_n$ be the ideal of V. Then $I(C)_n = I(V)_n$ for any $n \leq m$.*

Proof. Clearly $I(C)_m \subset I(V)_m$.Then we have to prove the opposite inclusion. Since $I(V) = \cap_{i=1}^s I(V_i)$ and $I(C) = \cap_{n=1}^s I(C_i)$, it is enough to prove the claim in the case of one irreducible curve C_i. Let $P_1, ..., P_{h_i}$, be the points of V_i. If $f \in I(V_i))_n$ then $f(\Phi_i(P_u)) = f(f_{i0}(P_u), ..., f_{ir}(P_u)) = 0$, for any $u = 1, ..., h_i$. Thus $f(f_{i0}, ..., f_{ir})$ is a polynomial of degree $nd_i \leq md_i$ vanishing on the $md_i + 1$ points P_i and then $f(f_1, ..., f_r) = 0$. But C_i is the smallest variety containing the points $[f_{i0}(a_0, a_1), ..., f_{ir}(a_0, a_1)]$, for any $[a_0, a_1] \in \mathbb{P}^1$ and then $C_i \subset V(f)$ that is $f \in I(C_i)$. □

Remark 2.3. *The h_i points of Lemma 2.2 are immediately found giving values $n.1_k$, $n \in \mathbb{Z}$, to the two parameters t_1, t_2.*

Theorem 2.4. *Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r_k$ be a non-degenerate curve such that C_i are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1_k \to \mathbb{P}^r_k$ given by forms of degree d_i , for any i. Let $d = \sum_{i=1}^s d_i$ and $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$ Consider $h_i = d_i(\alpha+1) + 1$ points of C_i for any i. Let V be the set of all these $h = \sum_{i=1}^s h_i = d(\alpha+1) + 1$ points and let $I(V) = \oplus_{n\geq 0} I(V)_n$ be the ideal of V. Set:*

$$W_{\alpha+1} = X_0 I(V)_\alpha + ... + X_r I(V)_\alpha \subset I(V)_{\alpha+1}$$

Then C has maximal rank if and only if

$H_V(\alpha - 1) = \binom{\alpha+r-1}{r}$, $H_V(\alpha) = d\alpha + s$

Moreover C is minimally generated if and only if

$$dim_k W_{\alpha+1} = Min\{(r+1)dim_k I(V)_\alpha, dim_k I(V)_{\alpha+1}\}$$

Proof. Follows immediately from Theorem 1.6, Definition 2.1 and Lemma 2.2. □

let $V = \{P_1, ..., P_h\} \in \mathbb{P}^r_k$ be a set of points . The vector space $I(V)_n$ is easily given by the null space of a matrix with elements in k. In fact if $R_n = \{f \in k[X_0, ..., X_r] \mid deg(f) = n\}$ then:

(2) $I(V)_n = \{f \in R_n \mid f(P_i) = 0,\ i = 1, ..., h\}$

If we denote with T_i, $i = 1, ..., u$, the terms of degree n in the indeterminates $X_0, ..., X_r$ then $S = \{T_1, ..., T_u\}$ is a basis of the k-vector space R_n. Let $G_n(V)$ be the following $\binom{n+r}{r} \times h$ matrix:

(3) $G_n(V) = (b_{ij})$ where $b_{ij} = T_i(P_j)$

If $f = a_1T_1 + ... + a_uT_u \in k[X_0, ..., X_r]$, we set

$$(f)_S = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix}$$

Then:

(4) $I(V)_n = \{f \in R_n \mid G_n(V)(f)_S = 0\}$

that is $f \in I(V)_n$ if and only if $(f)_S$ is a vector of the null space $N_n(V)$ of the matrix $G_n(V)$. This gives, by elementary linear algebra that:

(5) $dim_k(I(V)_n = \binom{n+r}{r} - rk(G_n(V))$ then $rk(G_n(V)) = H_V(n)$ $(rk = rank)$.

Let $\{(g_1)_S, ..., (g_e)_S\}$ be a basis of $N_n(V)$, $e = \binom{n+r}{r} - rk(G_n(V))$. then the $e(r+1)$ polynomials g_iX_j of $I(V)_{\alpha+1}$ span the vector space $W_{\alpha+1}$. Let:

(6) $M_{\alpha+1}(V)$ be the matrix whose columns are $(g_iX_j)_S$, for any i, j

then

(7) $dim_k W_{\alpha+1} = rk(M_{\alpha+1}(V))$

Corollary 2.5. *Let $C = \bigcup_{i=1}^{s} C_i \subset \mathbb{P}^r_k$ be a non-degenerate curve such that C_i are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1_k \to \mathbb{P}^r_k$ given by forms of degree d_i , for any i. Let $d = \sum_{i=1}^{s} d_i$ and $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$ Consider $h_i = d_i(\alpha + 1) + 1$ points of C_i for any i. Let V be the set of all these $h = \sum_{i=1}^{s} h_i = d(\alpha + 1) + 1$ points. Then the curves C_i are smooth and disjoint and C has maximal rank if and only if*

$rk(G_{\alpha-1}(V)) = \binom{\alpha+r-1}{r}$ and $rk(G_\alpha(V)) = d\alpha + s$.

Moreover, if C has maximal rank, C is minimally generated if and only if

$rk(M_{\alpha+1}(V)) = Min\{(r + 1)(\binom{\alpha+r}{r} - d\alpha - s), \binom{\alpha+r+1}{r} - d(\alpha + 1) - s)\}$

Proof. It follows immediately from Theorem 2.4 and (5), (7) $\square$

Corollary 2.6. *Let $C = \bigcup_{i=1}^{s} C_i \subset \mathbb{P}^r_k$ be a non-degenerate curve such that C_i are rational irreducible curves parametrized by maps $\Phi_i : \mathbb{P}^1_k \to \mathbb{P}^r_k$, $\Phi_i([t_1, t_2]) = [f_{i0}(t_1, t_2), ..., f_{ir}(t_1, t_2)]$, where $f_{ij} \in k[t_1, t_2]$ are homogeneous polynomials of the same degree d_i. Let $a_1, ..., a_q$ $(q = s(d + 1)(r + 1))$ be all the coefficients of all the polynomial $f_{ij}(t_1, t_2)$. Then:*

(a) the set $U \subset \mathbb{A}^q$ of the q-tuples $(a_1, ..., a_q)$ for which the curves C_i are smooth of degree d_i and disjoint, for any i, and C has maximal rank is open;

(b) the set $U' \subset \mathbb{A}^q$, of the q-tuples $(a_1,...,a_q) \in U$ for which C is minimally generated is open.

Proof. By construction (see also Remark 2.3) the matrices $G_{\alpha-1}(V)$, $G_\alpha(V)$ and $M_{\alpha+1}(V)$ have entries which are forms in $a_1,...,a_q$ and then their minors are forms in $a_1,...,a_q$. Moreover, by Proposition 1.3,(d) and (1),(5),(7), we have $rk(G_{\alpha-1}(V)) \leq c = \binom{\alpha+r-1}{r}$, $rk(G_\alpha(V)) \leq c' = d\alpha + s$ and $rk(M_{\alpha+1}(V)) \leq c" = Min\{(r+1)(\binom{\alpha+r}{r} - d\alpha - s), \binom{\alpha+r+1}{r} - d(\alpha+1) - s)\}$. Let $M_1,...,M_u$ be the minors of $G_{\alpha-1}(V)$ of order c, $M'_1,...,M'_v$ be the minors of $G_\alpha(V)$ of order c' and $M"_1,...,M"_w$ be the minors of $M_{\alpha+1}(V)$ of order $c"$. Consider the following closed sets of $\mathbb{A}^q$, $D: M_1 = 0,...,M_u = 0$, $D': M'_1 = 0,...,M'_v = 0$, $D": M"_1 = 0,...,M"_w = 0$ and let $U = \mathbb{A}^q_k - (D \cap D')$ and $U' = \mathbb{A}^q_k - (D \cap D' \cap D")$. If $(a_1,...,a_q) \in U$ then the matrices $G_{\alpha-1}(V)$ and $G_\alpha(V)$ have respectively rank c and c' and then C has maximal rank [Corollary 2.5]. If $(a_1,...,a_q) \in U'$ then the matrix $M_{\alpha+1}(V)$ has rank $c"$ and then C has maximal rank and is minimally generated [Corollary 2.5]. □

Remark 2.7. *For some fixed $r, s, d_1,...,d_s$ the open sets U and U' of Corollary 2.6 could be empty. For example the union C in $\mathbb{P}^3$ of two conics $C_1 \cup C_2$ is contained in a reducible quadric and C has not maximal rank. The union of 4 generic lines in $\mathbb{P}^3$ has maximal rank but it is non minimally generated* [9].

3. The ideal generation conjecture.

In this section we assume that the algebraically closed field k has characteristic zero.

Denote with $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$ the union of rational irreducible smooth disjoint curves C_i.

Consider the following two questions:

(a) *For what values of $r, s, d_1,...,d_s$ is it true that, for general $C_1,...,C_s$, the curve $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$ has maximal rank?*

(b) *For what values of $r, n, d_1,...,d_s$ is it true that, for general $C_1,...,C_s$, the curve $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r$ has maximal rank and is finitely generated?*

Clearly, by Corollary 2.6, to answer to one of the two questions for specific values of $r, s, d_1,...,d_s$, it is enough to find an example of curve C with maximal rank or with maximal rank and finitely generated. Question (a) had a complete answer with the following result.

Theorem 3.1. *Let $C = \bigcup_{i=1}^s C_i \subset \mathbb{P}^r_k$, $r \geq 3$, be the union of s rational irreducible smooth disjoint curves. Then, for general $C_1,...,C_s$, C has maximal rank, for any $r, s, d_1,...,d_s$, except the following s-uples $(d_1,...,d_s)$ of $\mathbb{P}^3_k$:*

$(2,2)$; $(2,4)$; $(2,2,2)$; $(2,2,4)$; $(2,2,2,2)$.

Proof. If $r \geq 4$, for general $C_1,...,C_s$ any C has maximal rank [4, Theorem 1].

If $r = 3$ the statement is proved in [1, Theorem 1] □

Question (b) has had a partial answer for lines and recently for irreducible curves in $\mathbb{P}^3$ with the following results:

Theorem 3.2. *(a) Let C be the union of s general lines in $\mathbb{P}^3$. Then C has maximal rank for any $s \neq 4$.*

(b) If C is a general irreducible rational curve of $\mathbb{P}^3$ of degree $d \neq 5$ then C has maximal rank and is minimally generated if $d \leq 73$ or $d \geq 932$.

Proof. (a) is proved in [9, Theorem 1]

(b) has been recently proved in [3, Theorem 0.2 and Remark 8.139] and in [10, Theorem 1] $\square$

In the following we want to show how the computer can be used to detect when a curve C is of maximal rank and minimally generated.

Let $\mathbb{Z}_p$ be the residue field of the integers modulo a prime p. If $\overline{f} = \overline{b}_1 T_1 + ... + \overline{b}_u T_u$ is a polynomial of $\mathbb{Z}_p[t_1, t_2]$ (T_i are the terms in t_1, t_2 and $\overline{b}_i \in \mathbb{Z}_p$ are the coefficents of $\overline{f}$) then $f = b_1 T_1 + ... + b_u T_u$ is the corresponding polynomial in $\mathbb{Z}[t_1, t_2]$.

Theorem 3.3. *Let $\overline{C} = \cup_i^s \overline{C}_i$ be the union of rational irreducible curves $C_i \subset \mathbb{P}^r_{\mathbb{Z}_p^*}$ ($\mathbb{Z}_p^*$= algebraic closure of $\mathbb{Z}_p^*$) represented by forms $\overline{f}_{ij} \in \mathbb{Z}_p[t_1, t_2]$, $j = 1, ..., r$. Let $C = \cup_i^s C_i$ be the union of the corresponding irreducible curves $C_i \subset \mathbb{P}^r_k$ represented by the forms $f_{ij} \in \mathbb{Z}[t_1, t_2] \subset k[t_1, t_2]$, of degree d_i. Let $d = \sum_{i=1}^s d_i$ and $\alpha = Min\{n \in \mathbb{N} \mid \binom{n+r}{r} > dn + s\}$ Suppose that $\overline{C}_i$ contains $h_i = d_i(\alpha + 1) + 1$ distinct points for any i. let $\overline{V}$ be the set of all these $h = \sum_{i=1}^s h_i = d(\alpha + 1) + s$ points. Consider the matrices* (3) *and* (6) *of section* 2 *relative to the set $\overline{V}$. If the following conditions hold:*

(a) $rk(G_{\alpha-1}(\overline{V})) = \binom{\alpha+r-1}{r}$;

(b) $rk(G_\alpha(\overline{V})) = d\alpha + s$;

(c) $rk(M_{\alpha+1}(\overline{V})) = Min\{(r+1)(\binom{\alpha+r}{r} - d\alpha - s), \binom{\alpha+r+1}{r} - d(\alpha+1) - s)\}$;

then the curves C_i are smooth and disjoint, C has maximal rank and is minimally generated.

Proof. It is enough to recall the construction of the matrices given in section 2 and to observe the following: if $Q = (a_{ij})$ is a matrix $m \times n$ with entries $a_{ij} \in \mathbb{Z}$ and $\overline{Q} = \overline{a}_{ij}$ is the matrix $m \times n$ whose entries are the integers a_{ij} modulo a prime p then $rk(\overline{Q}) \leq rk(Q)$. But, by Proposition 1.3,(d) and (1),(5),(7) of section 2, we have $rk(G_{\alpha-1}(V)) \leq \binom{\alpha+r-1}{r}$, $rk(G_\alpha(V)) \leq d\alpha + s$, and

$rk(M_{\alpha+1}(V)) \leq Min\{(r+1)(\binom{\alpha+r}{r} - d\alpha - s), \binom{\alpha+r+1}{r} - d(\alpha+1) - s)\}$.

Hence, by the assumptions (a),(b),(c), we have $rk(G_{\alpha-1}(\overline{V})) = rk(G_{\alpha-1}(V))$, $rk(G_\alpha(\overline{V})) = rk(G_\alpha(V))$ and $rk(M_{\alpha+1}(\overline{V})) = rk(M_{\alpha+1}(V))$ and the claim follows from Corollary 2.5. $\square$

Then, fixed $r, s, d_1, ..., d_s$, if we find an example of curve $\overline{C} = \cup_i^s \overline{C}_i$ for which the conditions $(a), (b), (c)$ of Theorem 3.3 are verified for a suitable prime p, then the corresponding curve $C \subset \mathbb{P}^r_k$, has maximal rank and is minimally generated.

Remark 3.4. *A key condition in Theorem* 3.3 *is to find $h_i = d_i(\alpha+1)+1$ distinct points on any $\overline{C}_i$ but if p is chosen large enough (e.g. $p = 31991$ has worked in all our examples) $\overline{C}_i$ always contains h_i distinct points (that are immediately found by giving numerical values to the parameters t_1, t_2 that are used to represent C_i, $i = 1, ..., s$).*

Using Theorem 3.3 it is easy to costruct an algorithm that for fixed $r, s, d_1, ..., d_s$ produces examples (if they exist) of curves C with maximal rank and minimally generated

Algorithm 3.5.

Step 1. *Input the integers* $r, s, d_1, ..., d_s$, *the prime* p *and the the coefficients (in* $\mathbb{Z}_p$*) of the polynomials* $\overline{f}_{ij}$ *that represent* $\overline{C}_i$.

Step 2. *Find* $h_i = d_i(\alpha+1)+1$ *distinct points of* C_i, *for any* i *(this can be done, if* p *is large enough, by giving numerical values to the parameters* t_1, t_2 *of* $\overline{f}_{ij}$*).*

Step 3. *Compute the ranks (with Gaussian reduction) of the matrices* $G_{\alpha-1}(\overline{V})$, $G_\alpha(\overline{V})$, $M_{\alpha+1}(\overline{V})$.

Step 4. *If these three ranks are respectively* $\binom{\alpha+r-1}{r}$, $d\alpha+s$, *and* $Min\{(r+1)(\binom{\alpha+r}{r}-d\alpha-s), \binom{\alpha+r+1}{r}-d(\alpha+1)-s)\}$ *then output: C has maximal rank and is minimally generated.*

Remark 3.6. *By Corollary* 2.6 *with a random choice of coefficients in Step*1 *one always finds a curve* C *of maximal rank and finitely generated unless for fixed* $r, s, d_1, ..., d_s$ *there is* <u>*no*</u> *curve* C *of maximal rank and finitely generated.*

Using the language C++, G. Albano and F. Cioffi have implemented Algorithm 3.5 on our PC (an Intel Pentium 200MHz with 40 MB RAM+ 100MB swap, running Linux 2.0).

The examples that we have run on the computer suggest the following:

Conjecture 3.7 [Ideal Generation Conjecture (I.G.C.) for rational curves]. *For every positive integers* $r, s, d_1, ..., d_s$, $r \geq 3$, *the union* $C = \bigcup_{i=1}^{s} C_i \subset \mathbb{P}_k^r$ *of general rational irreducible curves* C_i *of degree* d_i *has maximal rank and is minimally generated, except for the following s-uples* $(d_1, ..., d_s)$.

If $r = 3$: $(5); (1,2); (2,2); (2,3); (2,4); (3,4); (1,1,2); (1,2,2); (2,2,2); (2,2,3);$ $(2,2,4); (1,1,1,1); (1,2,2,2); (2,2,2,2); (1,2,2,2,2);$

If $r = 4$: $(5); (2,3); (1,1,2);$

If $r = 5$: $(3,4); (2,2,2);$

Remark 3.8. *With geometric arguments it is easily seen that for all the previous exceptionals s-tuples all the corresponding curves have not maximal rank or have maximal rank but are not minimally generated.*

In virtue of Corollary 2.6 to verify the conjecture it is enough to find an example for every s-uple $(d_1, ..., d_s)$ different from the exceptional ones. In particular we have verified by computer the following cases:

Theorem 3.9. *Let* $d = \sum_{i=1}^{s} d_i$ *be the degree of* C. *If* $s = 1$ *then the I.G.C. is true for* $d \leq 80$. *If* $s \leq 10$ *then the I.G.C. is true for* $d \leq 40$.

Remark 3.10. *Theoretically there is no limit on the degree d for producing examples of union of rational curves which satisfy the I.G.C. but when s and d get larger the computational time can be very high and the cases to treat very numerous.*

Remark 3.11. *Curves $C = \cup_{i=1}^{s} C_i$ which satisfy the I.G.C. could also be found by computing a minimal set of generators of $I(C)$ with Computer Algebra Systems based on Groebner Bases computation, like CoCoA. Using our PC, we have computed, with the last version of CoCoA* [5] (*that uses for the elimination of parameters th Hilbert Driven Algorithm of Traverso*), *a set of minimal generators of $I(C)$ for general rational curves C_i. With CoCoA one needs first to compute a minimal set of generators of the ideal of every curve C_i and then has to compute the generators of the intersection of all the ideals $I(C_i)$. This always needs much more computational time then with our algorithm. For example the minimal generators of the ideal of a general irreducible curve of degree* 40 *in* $\mathbb{P}^{15}$ *are found with CoCoA in* 222 *seconds and with our algorithm in* 88 *seconds. If the irreducible curve has degree* 80 *in* $\mathbb{P}^{10}$ *we have found the minimal generators in* 310 *seconds while with CoCoA we did not get any answer after* 9 *hours* (*then we have turned off the computer*). *The generators of the ideal of* 10 *curves with degree* 8 *in* $\mathbb{P}^{10}$ *are computed with CoCoA in* 3244 *seconds and with our algorithm in* 367 *seconds. Since to verify Theorem* 3.8 *on the computer one needs to check hundreds of cases it is clear that CoCoA is less suitable than our algorithm for checking I.G.C..*

References

[1] E. Ballico, *On the postulation of disjoint rational curves in a projective space*, Rend. Sem. Mat. Univer. Politecn. Torino **44** (1986), 46-52.

[2] E. Ballico, *Generators for the homogeneous ideal of s general points in* $\mathbb{P}^3$, J. Algebra **106** (1987), 46-52.

[3] E. Ballico, *On the minimal free resolution of non-special curves in* $\mathbb{P}^3$, Preprint.

[4] E. Ballico, P. Ellia, *On the Postulation of Many Disjoint Rational Curves in* $\mathbb{P}^n$*,* $n \geq 4$, Bollettino U.M.I. **(6) 4-B** (1985), 585-599.

[5] D. Capani, G. Niesi and L. Robbiano, *CoCoA (Computational Commutative Algebra)*, available at http://lancelot.dima.unige.it (1999).

[6] A.V. Geramita, F. Orecchia, *Minimally Generating Ideal Defining Certain Tangent Cones*, J. Algebra **78** (1982), 36-57.

[7] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a Theorem of Castelnuovo, and the Equations Defining Space Curves*, Invent. Math. **72** (1983), 491-506.

[8] A. Hirschowitz, C. Simpson, *La résolution minimale de l'idéal d'un arrangement général d'un grand nombre de points dans* $\mathbb{P}^n$, Invent. Math. **126** (1996), 467-503.

[9] M. Ida', *On the homogeneous ideal of the generic union of s-lines in* $\mathbb{P}^3$, J. Reine Angew. Math. **403** (1990), 67-153.

[10] M. Ida', *Generators for the generic rational space curve: low degree case*, Proceedings of the Ferrara Conference in honour of M. Fiorentini (1997).

[11] D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. studies **59** (1966).

Ferruccio Orecchia, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Complesso Universitario di Monte S. Angelo - Via Cintia , 80126 Napoli-Italy

E-mail address: `ORECCHIA@MATNA2.DMA.UNINA.IT`.