Computing Toric First Syzygies

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Introduction

Let $S \subset \mathbf{Z}^h$ be a finitely generated semigroup with zero element, such that $S \cap (-S) = \{0\}$. Let $\{n_1, \ldots, n_r\} \subset S$ be a set of generators for S.

Let k be a field, let $k[S]$ be the semigroup $k-$algebra associated to S, and let $R = k[X_1, \ldots, X_r]$ be the polynomial ring in r indeterminates.

$k[S]$ is obviously an $S-$graded ring,

$$k[S] = \oplus_{m \in S} k[S]_m,$$

where $k[S]_m = km$ and m is the symbol of $m \in S$ in $k[S]$. We also consider R as an $S-$graded ring, by assigning the degree n_i to X_i. We write $\mathbf{m} = \oplus_{m \in S - \{0\}} R_m$ for the irrelevant maximal ideal in R.

The $k-$algebra epimorphism,

$$\varphi : R \longrightarrow k[S],$$

defined by $\varphi(X_i) = n_i$, is a graded homomorphism of degree zero. Thus, the ideal $I_S = ker(\varphi)$ is a homogeneous ideal that it is called a *semigroup ideal* or *toric ideal* because it defines a toric variety.

By $S-$graded Nakayama's lemma (Proposition 1.4 in [2]), there exists a minimal free $S-$graded resolution of $k[S]$. Let N_i be the corresponding $i-$syzygies module ($N_0 = I$), and let $V_i(m) := \dfrac{(N_i)_m}{(\mathbf{m}N_i)_m}$, with $m \in S$.

A minimal generating set of N_i consists exactly of $dim_k(V_i(m))$ elements of degree m, for each m. In particular, since R is noetherian, one has $V_i(m) = 0$ for all m but finitely many values.

On the other hand, set $\Lambda = \{1, \ldots, r\}$ and $n_F := \sum_{i\in F} n_i\,(n_\emptyset = 0)$, for each $F \subset \Lambda$. One has an abstract simplicial complex given by

$$\Delta_m := \{F \subset \Lambda |\, m - n_F \in S\},$$

for each $m \in S$.

We denote by $\tilde{H}_i(\Delta_m)$ the $k-$vector spaces of the reduced homology. In [5] it is shown that $\tilde{H}_i(\Delta_m) \cong V_i(m)$, and in [3] a such isomorphism is explicity constructed.

Then, to find a minimal generating set for N_i it is enough to do the following (*Alg1*):

1. Find the set $C_i := \{m \in S|\, \tilde{H}_i(\Delta_m) \neq 0\}$.

2. For every $m \in C_i$, take the images of the basis elements for the homology spaces $\tilde{H}_i(\Delta_m)$ by the isomorphism above.

Since the set C_0 is arithmetically explicited in [4], this construction provides an algorithm for $i = 0$, [2].

We will find a finite set, $\mathcal{C}$, containing C_1. This set consists of the minimal $\mathbf{N}-$solutions to a finite number of linear systems over $\mathbf{Z}$, therefore it can be computed in an effective way, [6].

First of all, we are going to introduce different notations and definitions we will use in this paper.

We consider the $(h \times r)-$matrix

$$\mathcal{A} := (n_1|n_2|\cdots|n_r) \in \mathcal{M}_{h\times r}(\mathbf{Z}).$$

We denote $e_1, \ldots, e_r$ the canonical base of $\mathbf{N}^r$.

We want to find a finite set $\mathcal{C}$ to check possible elements in S corresponding to the degrees of the minimal first syzygies. This means that C_1 be contained in $\mathcal{C}$.

Definition 1 *Let $m \in S$ and $F := \{i_1, \ldots, i_t\} \subset \Lambda$ such that $1 \leq i_1 < \ldots < i_t \leq r$ and $\sharp F \geq 3$. We say F is a cavity of Δ_m if F verifies the following conditions:*

1. *$F_j \in \Delta_m$, $\forall j = 1, \ldots, t$ where*
$$F_j := \{i_j, i_{j+1}\},\ \forall j = 1, \ldots, t-1, \ \text{and } F_t := \{i_t, i_1\}.$$
2. *If $F_j \neq F' \subset F$, $\sharp F' \geq 2$, then $F' \notin \Delta_m$.*

We prove the following result.

Lemma 2 *Let $m \in S$ such that $\tilde{H}_1(\Delta_m) \neq 0$. Then there exists F cavity of Δ_m verifying $c_1 := \sum_{j=1}^{t} F_j$ is a generator of $\tilde{H}_1(\Delta_m)$.*

In particular, if F is a cavity, there exist $\alpha := (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in \mathbf{N}^{rt}$ such that
$$m = \mathcal{A}\alpha^{(1)} = \mathcal{A}\alpha^{(2)} = \cdots = \mathcal{A}\alpha^{(t)},$$
and $\alpha \gg e_F$ with $e_F := (e_{i_1} + e_{i_2}, e_{i_2} + e_{i_3}, \ldots, e_{i_{t-1}} + e_{i_t}, e_{i_t} + e_{i_1}) \in \mathbf{N}^{rt}$.

We denote by $\mathcal{A}(t)$ to the matrix
$$\mathcal{A}(t) := \begin{pmatrix} \mathcal{A} & -\mathcal{A} & 0 & 0 & 0 & & 0 & 0 \\ 0 & \mathcal{A} & -\mathcal{A} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{A} & -\mathcal{A} & 0 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \mathcal{A} & -\mathcal{A} \end{pmatrix} \in \mathcal{M}(\mathbf{Z})_{(t-1)h \times rt}.$$

Let R_F be the set of $\mathbf{N}-$solutions
$$R_F := \{\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in \mathbf{N}^{rt} |\, \mathcal{A}(t)\alpha = 0, \alpha \gg e_F\}.$$
We denote by vR_F the set of the minimal elements of R_F respect to the partial natural order over the uplas of non negative integers.

Let ΣR_F be the subset of S,
$$\Sigma R_F := \{p \in S |\, p = \mathcal{A}\alpha^{(1)} = \mathcal{A}\alpha^{(2)} = \cdots = \mathcal{A}\alpha^{(t)},\ (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in R_F\},$$
analogously we define ΣvR_F.

We are in conditions to prove next lemma,

Lemma 3 *If $\tilde{H}_1(\Delta_m) \neq 0$, then there exists F cavity of Δ_m verifying the lemma 2 and such that $m \in \Sigma v R_F$.*

One can compute vR_F in effective way, so we obtain an algorithm to compute $\mathcal{C}$.

Algorithm 4 Check Algorithm .-
In: *Set of generators $\{n_1, \ldots, n_r\}$ of a semigroup S.*
Out: *$\mathcal{C}$ set to check $\tilde{H}_1(\Delta_m) \neq 0$.*

1. *For i from 3 to r do*

   Compute $\mathcal{F} := \{F \subset \Lambda | \sharp F = i\}$.

   Compute vR_F, $\forall F \in \mathcal{F}$ (see [6]).

2. *$\mathcal{C} := \cup_{F \in \mathcal{F}} \Sigma v R_F$.*

Theorem 5 *The first syzygies of a toric variety can be determined using the construction in Alg1 and the algorithm 4 from the knowledge of the semigroup generators.*

As an application, by using the bounds in [7], we prove that the doubly-exponential degree in a minimal $1-$syzygy for binomial ideals (see [1]) can be improved for the subclass of toric ideals. A similar result for $i = 0$ appears in [9].

In [7], we can find some bounds to the minimal $\mathbf{N}-$solution of diophantine equations. The author considers the system of diophantine equations $Ax = 0$, where $rank(A) = s$. The author defines $||A||_{1,\infty} := sup_i\{\sum_j |A(i,j)|\}$, $||x||_1 := \sum_i |x_i|$, and $||M||_1 := sup_{x \text{ minimal solutions } Ax = 0} ||x||_1$. Pottier proves in [7] the following result.

Theorem 6

$$||M||_1 \leq (1 + ||A||_{1,\infty})^s.$$

In our particular case, we define

$$D := max_{F \subset \Lambda}\{D_F\} \text{ with } D_F := sup_i\{\sum_j |\mathcal{A}_F(i,j)|\} \in \mathbf{N},$$

and we prove the following theorem.

Theorem 7 *Let $(f^{(1)}, \ldots, f^{(p)})$ be an element of a minimal system of homogeneous generators for the first syzygy module of $k[S]$, then the total degree of $f^{(i)}$ is at most $(1+D)^{h(r-1)} + 2r$, $\forall i = 1, \ldots, p$.*

Let $A := (\mathcal{A} \,|\, -\mathcal{A})$ be a matrix in $\mathcal{M}_{h\times 2r}(\mathbf{Z})$. One can use this matrix to bound the generators of the ideal I_S (see [8]).

Corollary 8 *With the notations above, the total degree of an element of a minimal system of homogeneous generators for the first syzygy module of $k[S]$ is at most $(1+D)^{h(r-1)} + 2r + (1+||A||_{1,\infty})^h$.*

Therefore, this degree is singly-exponential in the number of variables.

A similar result can be obtained mixing [9] and [1].

Finally, we generalize all our results to the case of a semigroup with non trivial torsion, i.e. a commutative cancelative finitely generated semigroup

$$S \subset \mathbf{Z}^h \oplus \mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_s$$

with $S \cap (-S) = \{0\}$, and $a_1, \ldots a_s \in \mathbf{Z}$.

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