Functional Ideal Decomposition – a Tool for Kinematics

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The problem of finding general substitutions within systems of polynomial equations is investigated. For this purpose, the concept of functional ideal decomposition is introduced. In case of a 0-dimensional polynomial ideal over some coefficient field F we say that an ideal basis has a functional ideal decomposition („FID") if there are „outer decomposition factors" $g_i \in F[y_1, ..., y_n]$ and n common „inner decomposition factors" $h_j \in F[x_1, ..., x_n]$ s. t. all polynomials $f_i(x_1, ..., x_n)$ of the basis can be expressed as

$$f_i(x_1, ..., x_n) = g_i(h_1(x_1, ..., x_n), ..., h_n(x_1, ..., x_n))$$

and some nontriviality condition holds to exclude „unessential substitutions" (in this case: the number of common roots of the system of outer decomposition factors must be greater than 1 and smaller than the number of solutions of the original system).

A FID simplifies the (symbolic or iterative) solution of a given system significantly since it reduces the problem to the consecutive solution of two systems of considerably lower algebraic complexity. Note that all but one of the g_i or h_j may be trivial, thus yielding a family of interesting special cases.

FID generalizes the concept of functional decomposition of single polynomials and extends it to ideals in a similar way as primary decomposition extends the concept of factorization of polynomials to ideals. Note also the conceptual relationship of the presented problem with invariant theory.

First, a criterion will be presented for the existence of FID. The subsequent determination of inner and outer FID-factors is based on a repeated application of the Buchberger Algorithm and the identification of some subfield. The effort of some steps of the algorithm can be reduced through modular calculations. In case of polynomial systems with rational coefficients a large range of nontrivial problems can be solved with limited computational effort. Practical applications however require to determine FID-factors for systems with coefficients consisting of rational functions in some parameters. Currently, the latter task can only be achieved via interpolation. In case of systems with formal parameters, a FID not only simplifies the computation of solutions but may also yield a very compact representation of elimination ideals, of some final symbolic solution and of its univariate polynomial.

Kinematics seems to be a productive area of application of FID for various reasons:

- Industry needs *symbolic* solutions of systems of kinematic equations since iterative solution-techniques are far too slow to guide mechanism-effectors (e.g. robot hands) fast and smoothly along prescribed trajectories. The complexity of symbolic solutions is a major restriction for the design of innovative mechanisms. Industry does not produce manipulators in quantity unless simple symbolic solutions can be found. Any application of FID can be expected to reduce a solution of prohibitive complexity to a level which is sufficiently low to be used industrially.

- It was proven earlier that kinematic ideals of robots and fully parallel mechanisms are irreducible, i.e. factorization is of low importance in mechanism-kinematics. On the other hand, it is known that (ordinary) functional decomposition plays a significant role in kinematics and thus it can be expected that its generalization has significant applications also.
- Formal parameters usually appear with low degree in kinematic systems of equations and optimal ordinary solutions thereof. Consequently, it can be expected that interpolation techniques work successfully in practice once a FID is detected.
- FID will apply only to a limited class of mechanisms with a high degree of inherent regularity and structure. It is plausible that each member of this class will be of substantial theoretical interest for the mechanical engineer.

At this stage, no practical example of a mechanism admitting a FID is known. However, a complex industrial robot will be discussed which does not have a FID but only permits application of a closely related weaker simplification concept (to be called simultaneous functional decomposition). Due to this previously unknown simplification, the prohibitive algebraic complexity of the traditional symbolic solution can be reduced to a level which is acceptable for industrial use. This illustrates the potential scope and significance of the stronger concept of FID.