A Use of Computer Algebra for the Separability of the Perturbed Harmonic Oscillator with Homogeneous Polynomial Potentials *

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Abstract

In a previous paper of the author [1], the conjecture was posed on the Bertrand-Darboux integrability condition (BDIC), viewed from the Birkhoff-Gustavson (BG) normalization, for the perturbed harmonic oscillators (PHOs) with homogeneous polynomial potentials (HPPs). In this paper, the conjecture is refined along with the notion of separability, and is then proved to be true for the PHOs with HPPs of degree five and of degree seven through an effective use of computer algebra.

1. Introduction

It is widely known that the Birkhoff-Gustavson (BG) normalization has been utilized very effectively in various studies in Hamiltonian dynamical systems. The inverse problem of the BG normalization has been posed by the author as to *what kind of polynomial Hamiltonians can be brought into a given polynomial Hamiltonian in the Birkhoff-Gustavson normal form* [1-4]. An algorithm for solving the inverse problem has been given also [1,2], which fits computer algebra well.

As a very successful application of the inverse problem of the BG normalization, a new deep relation between the Bertrand-Darboux integrability condition (BDIC) of the perturbed harmonic oscillators (PHOs) with homogeneous polynomial potentials (HPPs) of degree-*three* and of degree-*four*. The BDIC-relation was conjectured to be true in [1] between the PHOs with HPPs of odd degree, say degree-r, and of degree-$2(r-1)$.

One of the aim of this paper is to present a refinement of the conjecture made from the separation-of-variables point of view:

Conjecture (refined) *The PHO with a HPP of degree-r ($r \geq 3$, odd) shares the same BG normal form up to degree-$2(r-1)$ with the PHO with a HPP of degree-$2(r-1)$ if and only if the PHO with the HPP of degree-r is separable within a rotation of Cartesian coordinates.*

Another aim of this paper is to report briefly that the above conjecture holds true in the case of $r = 5, 7$ besides the case of $r = 3$ proved in [1], through an effective use of computer algebra.

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The organization of this paper is outlined as follows. In Section 2, the ordinary and the inverse problems of the BG normalization is reviewed very briefly. Section 3 is devoted to the refinement of the conjecture. A $2 \times (r-2)$ matrix associated with a given HPP of degree-r will play a key role; the rank-one condition for that matrix turns out to be equivalent to the separability of a given PHO within a rotation of Cartesian coordinates. In Section 4, the rank-one condition for the matrix associated with HPPs of degree-r is shown to be a necessary condition for the Conjecture to be true for $r = 5, 7$ with an effective use of computer algebra, REDUCE 3. 7. Section 5 is devoted to the proof of the sufficiency. Section 6 is for concluding remarks.

2. The ordinary and the inverse problems of the BG normalization

In this section, the ordinary and the inverse problems of the BG normalization is reviewed very briefly (see [1,2] for more detail). For simplicity, our discussion will be restricted only to the two-degree-of-freedom systems in with a semisimple 1-1 resonant equilibrium point (see [5], for detail).

To make the review compact, let us prepare the following polynomials of degree-ρ ($\rho \geq$ 4:even);

$$\begin{aligned} J(q,p) &= \frac{1}{2}\sum_{j=1}^{2}(p_j^2+q_j^2), & K(q,p) &= J(q,p) + \sum_{k=3}^{\rho} K_k(q,p), \\ G(\xi,\eta) &= J(\xi,\eta) + \sum_{k=3}^{\rho} G_k(\xi,\eta), & H(q,p) &= J(q,p) + \sum_{k=3}^{\rho} H_k(q,p), \\ W(q,\eta) &= \sum_{j=1}^{N} q_j\eta_j + \sum_{k=3}^{\rho} W_k(q,\eta), & S(q,\eta) &= -\sum_{j=1}^{N} q_j\eta_j - \sum_{k=3}^{\rho} S_k(q,\eta), \end{aligned} \tag{2.1}$$

where (q,p) and (ξ,η) stand for canonical coordinates in $\mathbf{R}^2 \times \mathbf{R}^2$, K_k, G_k, H_k, W_k and S_k are homogeneous polynomials of degree-k in their own variables.

Definition 1 *The polynomial G is assumed to be in BG normal form (BGNF) <u>up to degree-ρ</u>, if and only if G satisfies the Poisson-commuting relation,*

$$\{J(\xi,\eta), G_k(\xi,\eta)\} = \sum_{j=1}^{2}\left(\xi_j\frac{\partial G_k}{\partial \eta_j} - \eta_j\frac{\partial G_k}{\partial \xi_j}\right) = 0 \quad (k = 3, \cdots, \rho), \tag{2.2}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket in (ξ,η).

To define the ordinary and the inverse problems in a mathematically rigorous way, we make the following assumption on W and S:

Assumption 2 [1] *The polynomials W and S satisfy*

$$W, S \in imageD_{q,\eta} \quad with \quad D_{q,\eta} = \sum_{j=1}^{2}\left(q_j\frac{\partial}{\partial \eta_j} - \eta_j\frac{\partial}{\partial q_j}\right). \tag{2.3}$$

After the preparation above, the ordinary and the inverse problems of the BG normalization of degree-ρ can be summarized to as presented in Table 1.

[1] The assumption on W ensures the uniqueness of the BG normal form [1].

	the ordinary problem	the inverse problem
given	$K(q,p)$	$G(\xi,\eta)$: BGNF
identify	$G(\xi,\eta)$: BGNF	$H(q,p)$
canonical transformation [a]	$(q,p)\to(\xi,\eta)$	$(\xi,\eta)\to(q,p)$
generating function[b]	$W(q,\eta)$: 2nd-type	$S(q,\eta)$: 3rd-type
defining equation [c]	$G(\frac{\partial W}{\partial\eta},\eta)=K(q,\frac{\partial W}{\partial q})$	$H(q,-\frac{\partial S}{\partial q})=G(-\frac{\partial S}{\partial\eta},\eta)$

Table 1 Summary of the ordinary and the inverse problems

Remarks on Table 1 (a) The transformations might work only around the origin. (b) A generating function of the old position variables and the new momentum one (resp. the old momentum and the new position ones) is referred to as of the 2nd (resp. 3rd) type (see [6]). (c) The equality is taken up to *degree-ρ*.

As a test case, we take the one-parameter Hénon-Heiles (HH) system which has been repeatedly studied as a typical non-linear Hamiltonian system [1]. To solve both the ordinary and the inverse problem for the HH system, the author has used the program named ANFER on REDUCE3.7 developed by himself (for a prototype, visit [7]). We start with the one-parameter Hénon-Heiles Hamiltonian

$$K_\mu(q,p)=\frac{1}{2}\sum_{j=1}^{2}(p_j^2+q_j^2)+q_1^2q_2+\mu q_2^3 \qquad (\mu\in\mathbf{R}). \tag{2.4}$$

Up to degree-4, $K_\mu(q,p)$ is converted into the BG normal form Hamiltonian,

$$\begin{aligned}G_\mu(\xi,\eta)&=\frac{1}{2}(\zeta_1\overline{\zeta}_1+\zeta_2\overline{\zeta}_2)\\&+\frac{1}{48}\Big\{-5\zeta_1^2\overline{\zeta}_1^2-45\mu^2\zeta_2^2\overline{\zeta}_2^2-(8+36\mu)\zeta_1\zeta_2\overline{\zeta}_1\overline{\zeta}_2\\&\qquad+3\mu\zeta_1^2\overline{\zeta}_2^2+3\mu\zeta_2^2\overline{\zeta}_1^2-6\zeta_1^2\overline{\zeta}_2^2-6\zeta_2^2\overline{\zeta}_1^2\Big\},\end{aligned} \tag{2.5}$$

where $\zeta_j=\xi_j+i\eta_j$ $(j=1,2)$. The BG normal form Hamiltonian $G(\xi,\eta)$ is inverted, in turn, up to degree-4 to identify the class of Hamiltonians of the following form; on setting $z_j=q_j+ip_j$ $(j=1,2)$, the resultant Hamiltonian $H_\mu(q,p)$ is calculated through ANFER to be

$$H_\mu(q,p)=\frac{1}{2}\sum_{j=1}^{2}(p_j^2+q_j^2)+H_{\mu,3}(q,p)+H_{\mu,4}(q,p) \tag{2.6}$$

with

$$\begin{aligned}H_{\mu,3}(q,p)&=a_1z_1^3+a_2z_1^2z_2+a_3z_1z_2^2+a_4z_2^3+a_5z_1^2\overline{z}_1\\&+a_6z_1^2\overline{z}_2+a_7z_1z_2\overline{z}_1+a_8z_1z_2\overline{z}_2+a_9z_2^2\overline{z}_1+a_{10}z_2^2\overline{z}_2\\&+\overline{a}_1\overline{z}_1^3+\overline{a}_2\overline{z}_1^2\overline{z}_2+\overline{a}_3\overline{z}_1\overline{z}_2^2+\overline{a}_4\overline{z}_2^3+\overline{a}_5z_1\overline{z}_1^2\\&+\overline{a}_6z_2\overline{z}_1^2+\overline{a}_7z_1\overline{z}_1\overline{z}_2+\overline{a}_8z_2\overline{z}_1\overline{z}_2+\overline{a}_9z_1\overline{z}_2^2+\overline{a}_{10}z_2\overline{z}_2^2\end{aligned} \tag{2.7a}$$

and

$$
\begin{aligned}
H_{\mu,4}(q,p) = {} & c_1 z_1^4 + c_2 z_1^3 z_2 + c_3 z_1^2 z_2^2 + c_4 z_1 z_2^3 + c_5 z_2^4 + c_6 z_1^3 \overline{z}_1 + c_7 z_1^3 \overline{z}_2 \\
& + c_8 z_1^2 z_2 \overline{z}_1 + c_9 z_1^2 z_2 \overline{z}_2 + c_{10} z_1 z_2^2 \overline{z}_1 + c_{11} z_1 z_2^2 \overline{z}_2 + c_{12} z_2^3 \overline{z}_1 + c_{13} z_2^3 \overline{z}_2 \\
& + \overline{c}_1 \overline{z}_1^4 + \overline{c}_2 \overline{z}_1^3 \overline{z}_2 + \overline{c}_3 \overline{z}_1^2 \overline{z}_2^2 + \overline{c}_4 \overline{z}_1 \overline{z}_2^3 + \overline{c}_5 \overline{z}_2^4 + \overline{c}_6 z_1 \overline{z}_1^3 + \overline{c}_7 z_2 \overline{z}_1^3 \\
& + \overline{c}_8 z_1 \overline{z}_1^2 \overline{z}_2 + \overline{c}_9 z_2 \overline{z}_1^2 \overline{z}_2 + \overline{c}_{10} z_1 \overline{z}_1 \overline{z}_2^2 + \overline{c}_{11} z_2 \overline{z}_1 \overline{z}_2^2 + \overline{c}_{12} z_1 \overline{z}_2^3 + \overline{c}_{13} z_2 \overline{z}_2^3 \\
& + 8 \left(a_9 \overline{a}_{10} z_2^2 \overline{z}_1 \overline{z}_2 + a_9 \overline{a}_9 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_{10} \overline{a}_9 z_1 z_2 \overline{z}_2^2 \right. \\
& \qquad \left. + a_5 \overline{a}_6 z_1 z_2 \overline{z}_1^2 + a_6 \overline{a}_5 z_1^2 \overline{z}_1 \overline{z}_2 + a_6 \overline{a}_6 z_1 z_2 \overline{z}_1 \overline{z}_2 \right) \\
& + 6 \left(a_1 \overline{a}_1 z_1^2 \overline{z}_1^2 + a_{10} \overline{a}_{10} z_2^2 \overline{z}_2^2 + a_4 \overline{a}_4 z_2^2 \overline{z}_2^2 + a_5 \overline{a}_5 z_1^2 \overline{z}_1^2 \right) \\
& + 4 \left(a_8 \overline{a}_5 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_8 \overline{a}_6 z_2^2 \overline{z}_1 \overline{z}_2 + a_8 \overline{a}_9 z_1^2 \overline{z}_2^2 + a_9 \overline{a}_7 z_1 z_2 \overline{z}_1^2 \right. \\
& \qquad + a_9 \overline{a}_8 z_2^2 \overline{z}_1^2 + a_1 \overline{a}_2 z_1^2 \overline{z}_1 \overline{z}_2 + a_{10} \overline{a}_7 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_2 \overline{a}_1 z_1 z_2 \overline{z}_1^2 \\
& \qquad + a_3 \overline{a}_4 z_1 z_2 \overline{z}_2^2 + a_4 \overline{a}_3 z_2^2 \overline{z}_1 \overline{z}_2 + a_5 \overline{a}_8 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_6 \overline{a}_7 z_1^2 \overline{z}_2^2 \\
& \qquad \left. + a_6 \overline{a}_8 z_1 z_2 \overline{z}_2^2 + a_7 \overline{a}_{10} z_1 z_2 \overline{z}_1 \overline{z}_2 + a_7 \overline{a}_6 z_2^2 \overline{z}_1^2 + a_7 \overline{a}_9 z_1^2 \overline{z}_1 \overline{z}_2 \right) \\
& + \tfrac{8}{3} \left(a_2 \overline{a}_2 z_1 z_2 \overline{z}_1 \overline{z}_2 + a_3 \overline{a}_3 z_1 z_2 \overline{z}_1 \overline{z}_2 \right) \\
& + 2 \left(a_8 \overline{a}_{10} z_1 z_2 \overline{z}_2^2 + a_8 \overline{a}_7 z_1^2 \overline{z}_1 \overline{z}_2 + a_8 \overline{a}_7 z_1 z_2 \overline{z}_2^2 + a_8 \overline{a}_8 z_2^2 \overline{z}_2^2 \right. \\
& \qquad + a_1 \overline{a}_3 z_1^2 \overline{z}_2^2 + a_{10} \overline{a}_8 z_2^2 \overline{z}_1 \overline{z}_2 + a_2 \overline{a}_4 z_1^2 \overline{z}_2^2 + a_3 \overline{a}_1 z_2^2 \overline{z}_1^2 \\
& \qquad + a_4 \overline{a}_2 z_2^2 \overline{z}_1^2 + a_5 \overline{a}_7 z_1^2 \overline{z}_1 \overline{z}_2 + a_7 \overline{a}_5 z_1 z_2 \overline{z}_1^2 + a_7 \overline{a}_7 z_1^2 \overline{z}_1^2 \\
& \qquad + a_7 \overline{a}_8 z_1 z_2 \overline{z}_1^2 + a_7 \overline{a}_8 z_2^2 \overline{z}_1 \overline{z}_2 - a_8 \overline{a}_6 z_1 z_2 \overline{z}_1^2 - a_7 \overline{a}_9 z_1 z_2 \overline{z}_2^2 \\
& \qquad - a_9 \overline{a}_5 z_2^2 \overline{z}_1^2 - a_9 \overline{a}_7 z_2^2 \overline{z}_1 \overline{z}_2 - a_9 \overline{a}_9 z_2^2 \overline{z}_2^2 - a_{10} \overline{a}_6 z_2^2 \overline{z}_1^2 \\
& \qquad \left. - a_5 \overline{a}_9 z_1^2 \overline{z}_2^2 - a_6 \overline{a}_{10} z_1^2 \overline{z}_2^2 - a_6 \overline{a}_6 z_1^2 \overline{z}_1^2 - a_6 \overline{a}_8 z_1^2 \overline{z}_1 \overline{z}_2 \right) \\
& + \tfrac{4}{3} \left(+ a_2 \overline{a}_3 z_1^2 \overline{z}_1 \overline{z}_2 + a_2 \overline{a}_3 z_1 z_2 \overline{z}_2^2 + a_3 \overline{a}_2 z_1 z_2 \overline{z}_1^2 + a_3 \overline{a}_2 z_2^2 \overline{z}_1 \overline{z}_2 \right) \\
& + \tfrac{2}{3} \left(a_2 \overline{a}_2 z_1^2 \overline{z}_1^2 + a_3 \overline{a}_3 z_2^2 \overline{z}_2^2 \right) \\
& + \tfrac{1}{48} \left(-8 z_1 z_2 \overline{z}_1 \overline{z}_2 - 5 z_1^2 \overline{z}_1^2 - 6 z_1^2 \overline{z}_2^2 - 6 z_2^2 \overline{z}_1^2 \right. \\
& \qquad \left. -36 \mu z_1 z_2 \overline{z}_1 \overline{z}_2 - 45 \mu^2 z_2^2 \overline{z}_2^2 + 3 \mu z_1^2 \overline{z}_2^2 + 3 \mu z_2^2 \overline{z}_1^2 \right),
\end{aligned}
\tag{2.7b}
$$

where a_h ($h = 1, \cdots, 10$) and c_ℓ ($\ell = 1, \cdots, 13$) are complex-valued parameters chosen arbitrarily. One might understand, from the number of arbitrarily chosen parameters ($2 \times (10 + 13) = 46$ in real valued expression !), an effectiveness of computer algebra in the inverse problem.

3. Conjecture refined from the separability-viewpoint

In the paper [1], the ordinary and the inverse problems of the BG normalization are applied to the perturbed harmonic oscillators (PHOs) with homogeneous polynomial potentials (HPPs) of degree-3, Studying what kind of PHO with HPP of *degree*-3 can share its BG normal form with the PHO with a HPP of *degree*-4, we have the following theorems [1]:

Theorem 3 [1] *The PHO Hamiltonian with a HPP of degree-*3 *shares its BG normal from up to degree-*4 *with the PHO Hamiltonian with a HPP of degree-*4 *if and only if the PHO Hamiltonian of degree-*3 *satisfies the Bertrand-Darboux integrability condition (BDIC) of generic case* [2] .

Theorem 4 [1] *The PHO Hamiltonian with a HPP of degree-*3 *and the PHO Hamiltonian with a HPP of degree-*4 *share the same BG normal form up to degree-*4 *if the Hamiltonians of both oscillators satisfy the Bertrand-Darboux integrability condition (BDIC) of generic case* [2].

In this section, we study a more precise meaning of the BDIC of *generic case* for the PHOs with HPPs of an arbitrary degree: The genericity will turn out to be equivalent to the separability of the PHO Hamiltonians within a rotation. To show this, we start with reviewing the very old theorem due to Darboux and Bertrand [8].

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[2] We here refer eq.(54a) in [1] for the PHOs of degree-3 and (55b) in [1] for of degree-4 to as the BDIC of generic case.

Theorem 5 (Bertrand-Darboux) [8] *Let F be a natural Hamiltonian of the form,*

$$F(q,p) = \frac{1}{2}\sum_{j=1}^{2} p_j^2 + V(q), \tag{3.1}$$

where $V(q)$ a differentiable function in q. Then, the following three statements are equivalent for the Hamiltonian system with F.

(1) *There exists a set of real-valued constants, $(\alpha,\beta,\beta',\gamma,\gamma') \neq (0,0,0,0,0)$, for which $V(q)$ satisfies*

$$\begin{aligned}\left(\frac{\partial^2 V}{\partial q_2^2} - \frac{\partial^2 V}{\partial q_1^2}\right)&(-2\alpha q_1 q_2 - \beta' q_2 - \beta q_1 + \gamma) \\ &+2\frac{\partial^2 V}{\partial q_1 \partial q_2}(\alpha q_2^2 - \alpha q_1^2 + \beta q_2 - \beta' q_1 + \gamma') \\ &\quad + \frac{\partial V}{\partial q_1}(6\alpha q_2 + 3\beta) - \frac{\partial V}{\partial q_2}(6\alpha q_1 + 3\beta') = 0.\end{aligned} \tag{3.2}$$

(2) *The Hamiltonian system with F admits an integral of motion quadratic in momenta.*

(3) *The Hamiltonian F is separable in either Cartesian, polar, parabolic or elliptic coordinates.*

Due to the statement (2) in Theorem 5, a natural Hamiltonian system with F is always integrable if (3.2) holds true. In this regard, equation (3.2) is referred to as the Bertrand-Darboux integrability condition (BDIC).

We wish to draw another expression of the BDIC for the PHOs with HPPs of arbitrary degrees. Let a HPP, denoted by $V^{(r)}(q)$, of degree-r be written in the form

$$V^{(r)}(q) = \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} f_j q_1^j q_2^{r-j} \qquad (f_j \in \mathbf{R},\ j = 0,1,\cdots,r), \tag{3.3}$$

where $\begin{pmatrix} r \\ j \end{pmatrix}$ denotes the binomial coefficient. After a long but straightforward calculation of (3.2) with $V(q) = (1/2)(q_1^2 + q_2^2) + V^{(r)}(q)$, we have the following.

Theorem 6 *For $V^{(r)}(q)$, a HPP of degree-r given by (3.3), let the $2 \times (r-1)$ matrix $\mathcal{B}(V^{(r)})$ be defined by*

$$\mathcal{M}(V^{(r)}) = \begin{pmatrix} f_0 - f_2 & f_1 - f_3 & \cdots & f_{r-2} - f_r \\ f_1 & f_2 & \cdots & f_{r-1} \end{pmatrix}. \tag{3.4}$$

Then the BDIC for the PHO with $V^{(r)}$ is equivalent to either of the following cases:

(1) *In the case of odd $r \geq 5$,*

$$\operatorname{rank} \mathcal{M}(V^{(r)}) = 1. \tag{3.5}$$

(2) *In the case of even $r \geq 4$, either of the follwoing: (3.6a) or (3.6b);*

$$\operatorname{rank} \mathcal{M}(V^{(r)}) = 1. \tag{3.6a}$$

$$\begin{cases} f_{2j} = \dfrac{2j-1}{r-2j+1} f_{2(j-1)} \quad \text{with} \quad f_0 \neq 0 \\ f_{2j-1} = 0 \end{cases} \qquad (j = 1,\cdots,\frac{r}{2}). \tag{3.6b}$$

(3) *In the case of $r = 3$, either of the following: (3.7a), (3.7b) or (3.7c);*

$$\text{rank}\,\mathcal{M}(V^{(3)}) = 1. \tag{3.7a}$$

$$f_0 = 6f_2 \neq 0, \quad f_1 = f_3 = 0. \tag{3.7b}$$

$$f_3 = 6f_1 \neq 0, \quad f_0 = f_2 = 0. \tag{3.7c}$$

It is worth noting here that the BDIC of generic case for the PHOs with HPPs of degree-3 in [1] is equivalent to (3.7a) and that of degree-4 in [1] to (3.6a) [3] .. The expressions (3.5)-(3.7c) of the BDIC for the PHOs can be interpreted from the separability viewpoint as follows:

Theorem 7 *The expressions (3.5)-(3.7c) of the BDIC condition for the PHOs with HPPs are equivalent to the separability of the PHO Hamiltonians in either of the following coordinates:*

- *(3.5), (3.6a) and (3.7a)* : *separable within rotations of Cartesian coordinates*,
- *(3.6b)* : *separable in polar coordinates*, (3.8)
- *(3.7b) and (3.7c)* : *separable in parabolic coordinates.*

From Theorem 6, one can easily see that the expression of the BDIC common among the PHOs with HPPs of arbitrary degrees is the rank-one condition of the matrix $\mathcal{M}(V^{(r)})$ associated with $V^{(r)}(q)$. Hence, on taking Theorem 7 into account, the conjecture posed in [1] can be refined as to Conjecture given in Sec.1 in terms of the separability in Cartesian coordinates.

4. Separability as a necessity in Conjecture

In this section, we report that the separability indeed turns out to be a necessary condition in Conjecture for $r = 5, 7$ with an effective use of computer algebra, although the proof is not yet completed for general odd r.

To prove the necessity, let us consider the situation that the PHO Hamiltonian,

$$\begin{aligned} K^{(r)}(q,p) &= \frac{1}{2}\sum_{k=1}^{2}(p_k^2 + q_k^2) + V^{(r)}(q), \\ V^{(r)}(q) &= \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} f_j q_1^j q_2^{r-j}, \end{aligned} \tag{4.1}$$

with a HPP of degree-r shares the same BG normal form with the PHO Hamiltonian,

$$\begin{aligned} K^{(2(r-1))}(q,p) &= \frac{1}{2}\sum_{k=1}^{2}(p_k^2 + q_k^2) + V^{(2(r-1))}(q), \\ V^{(2(r-1))}(q) &= \sum_{j=0}^{2(r-1)} \begin{pmatrix} 2(r-1) \\ j \end{pmatrix} g_j q_1^j q_2^{2(r-1)-j}, \end{aligned} \tag{4.2}$$

with a HPP of degree-$2(r-1)$. Throughout this section, the integer $r \geq 3$ is assumed to be odd.

[3] From this expression, the expression due to Yamaguchi and Nambu [9] turns out to be redundant.

Along with the algorithm of solving the ordinary problem [1,2], the BG normal forms $G^{(r)}$ and $G^{(2(r-1))}$ up to degree-$2(r-1)$ for $K^{(r)}$ and $K^{(2(r-1))}$ are calculated, respectively, to be

$$
\begin{aligned}
G^{(r)}(\xi,\eta) = & \frac{1}{2}\sum_{k=1}^{2}(\eta_k^2+\xi_k^2) \\
& + \left[\frac{1}{2}\sum_{k=1}^{2}\left\{\left(\frac{\partial}{\partial q_j}((D'_{q,\eta})^{-1}V^{(r)})\right)^2\right.\right. \\
& \left.\left. - \left(\frac{\partial}{\partial \eta_j}((D'_{q,\eta})^{-1}V^{(r)})\right)^2\right\}\right]^{\ker}\Bigg|_{q=\xi},
\end{aligned}
\tag{4.3}
$$

and

$$
G^{(2(r-1))}(\xi,\eta) = \frac{1}{2}\sum_{k=1}^{2}(\eta_k^2+\xi_k^2) + \left[V^{(2(r-1))}\right]^{\ker}\Big|_{q=\xi}, \tag{4.4}
$$

where $D'_{q,\eta}$ stands for the restriction of $D_{q,\eta}$ defined by (2.3) on image$D_{q,\eta}$ and the superscript ker indicates the $\ker D_{q,\eta}$-component of the polynomials enclosed by the square brackets.

Since the homogeneous polynomials of degree-$2(r-1)$ in $\ker D_{q,\eta}$ are spanned by the monomials,

$$
\zeta_1^{\ell}\zeta_2^{r-1-\ell}\overline{\zeta}_1^{m-\ell}\overline{\zeta}_2^{r-1-m+\ell} \quad (\zeta_j = \xi_j + i\eta_j (j=1,2)) \tag{4.5a}
$$

subject to

$$
0 \le m \le 2(r-1), \quad \max(0, m-(r-1)) \le \ell \le \min(r-1, m), \tag{4.5b}
$$

what we have to do is to equate the coefficient of each the monomials $\zeta_1^{\ell}\zeta_2^{r-1-\ell}\overline{\zeta}_2^{m-\ell}\overline{\zeta}_2^{r-1-m+\ell} \in \ker D_{q,\eta}$ arising from $G^{(r)}$ and the one from $G^{(2(r-1))}$. On denoting by $\mathcal{C}^{(r)}_{m,\ell}$ the coefficient of the monomial $\zeta_1^{\ell}\zeta_2^{r-1-\ell}\overline{\zeta}_2^{m-\ell}\overline{\zeta}_2^{r-1-m+\ell}$ in $G^{(r)}$ and by $\mathcal{C}^{(2(r-1))}_{m,\ell}$ in $G^{(2(r-1))}$, we obtain the equations

$$
\mathcal{C}^{(r)}_{m,1}\mathcal{C}^{(2(r-1))}_{m,0} = \mathcal{C}^{(r)}_{m,0}\mathcal{C}^{(2(r-1))}_{m,1} \quad (m = 2, 3, \cdots, 2r-4), \tag{4.6}
$$

as a necessary condition for Conjecture to be valid.

By a further calculation, we see that $\mathcal{C}^{(2(r-1))}_{m,\ell}$s are expressed to be

$$
\mathcal{C}^{(2(r-1))}_{m,\ell} = 4^{1-r} g_m \begin{pmatrix} 2(r-1) \\ m \end{pmatrix}\begin{pmatrix} m \\ \ell \end{pmatrix}\begin{pmatrix} 2(r-1)-m \\ r-1-\ell \end{pmatrix}, \tag{4.7}
$$

and $\mathcal{C}^{(r)}_{m,\ell}$s to be

$$
\begin{aligned}
\mathcal{C}^{(r)}_{m,\ell} = & \frac{2r^2}{4^r}\sum_{j=\mu_J}^{M_J}\begin{pmatrix} r-1 \\ j \end{pmatrix}\begin{pmatrix} r-1 \\ m-j \end{pmatrix}(f_j f_{m-j} + f_{j+1} f_{m-j+1}) \\
& \times \sum_{k=\mu_K}^{M_K}\sum_{h=\mu_H}^{M_H}\frac{\begin{pmatrix} j \\ k \end{pmatrix}\begin{pmatrix} r-1-j \\ h \end{pmatrix}\begin{pmatrix} m-j \\ \ell-k \end{pmatrix}\begin{pmatrix} r-1-(m-j) \\ r-1-\ell-h \end{pmatrix}}{\{(2(k+h+1)-r\}\{2(k+h)-r\}},
\end{aligned}
\tag{4.8a}
$$

where

$$
\begin{aligned}
\mu_J &= \max(0, m-(r-1))), & M_J &= \min(m, r-1), \\
\mu_K &= \max(0, \ell-(m-j)), & M_K &= \min(j, \ell), \\
\mu_H &= \max(0, (m-j)-\ell), & M_J &= \min(r-1-\ell, r-1-j).
\end{aligned}
\tag{4.8b}
$$

However, matters are not easy yet, because the expression (4.8) is not explicit enough to draw (3.5) from (4.6). The condition (3.5) is drawn successfully only in the cases of $r = 5, 7$ besides the case of $r = 3$ with an effective use of computer algebra. In the case of $r = 5$, computer algebra reduces eq.(4.6) to

$$\begin{aligned} &\begin{vmatrix} f_0 - f_2 & f_1 - f_3 \\ f_1 & f_2 \end{vmatrix} = 0 \quad (m = 2), \qquad \begin{vmatrix} f_0 - f_2 & f_2 - f_4 \\ f_1 & f_3 \end{vmatrix} = 0 \quad (m = 3), \\ &\begin{vmatrix} f_0 - f_2 & f_3 - f_5 \\ f_1 & f_4 \end{vmatrix} - 45 \begin{vmatrix} f_1 - f_3 & f_2 - f_4 \\ f_2 & f_3 \end{vmatrix} = 0 \quad (m = 4), \end{aligned} \tag{4.9}$$

which is indeed equivalent to (3.5) with $r = 5$. Similarly, in the case of $r = 5$, computer algebra reduces (4.6) to

$$\begin{aligned} &\begin{vmatrix} f_0 - f_2 & f_1 - f_3 \\ f_1 & f_2 \end{vmatrix} = 0 \quad (m = 2), \qquad \begin{vmatrix} f_0 - f_2 & f_2 - f_4 \\ f_1 & f_3 \end{vmatrix} = 0 \quad (m = 3), \\ &\begin{vmatrix} f_0 - f_2 & f_3 - f_5 \\ f_1 & f_4 \end{vmatrix} + \tfrac{105}{31} \begin{vmatrix} f_1 - f_3 & f_2 - f_4 \\ f_2 & f_3 \end{vmatrix} = 0 \quad (m = 4), \\ &\begin{vmatrix} f_0 - f_2 & f_4 - f_6 \\ f_1 & f_5 \end{vmatrix} - 35 \begin{vmatrix} f_1 - f_3 & f_3 - f_5 \\ f_2 & f_4 \end{vmatrix} = 0 \quad (m = 5), \\ &\begin{vmatrix} f_0 - f_2 & f_5 - f_7 \\ f_1 & f_6 \end{vmatrix} + 105 \begin{vmatrix} f_1 - f_3 & f_4 - f_6 \\ f_2 & f_5 \end{vmatrix} - 3500 \begin{vmatrix} f_2 - f_4 & f_3 - f_5 \\ f_3 & f_4 \end{vmatrix} = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (m = 6), \end{aligned} \tag{4.10}$$

which is equivalent again to (3.5) with $r = 7$. To summarize, we have the following .

Theorem 8 *In the cases of $r = 5, 7$, if the PHO Hamiltonian with HPP of degree-r shares the same BG normal form up to degree-$2(r-1)$ with the PHO Hamiltonian with a HPP of degree-$2(r-1)$ then the PHO Hamiltonian of degree-r is separable within a rotation of Cartesian coordinates.*

5. Separability as a sufficiency in Conjecture

In this section, we show that the separability of the PHOs with HPPs within rotations of Cartesian coordinates is a sufficient condition. The equivariance of the rotational transformation and the BG normalization will play a key role in the proof.

Let R be a 2×2 rotational matrix fixed arbitrarily, which provides the pair of canonical transformations,

$$\phi_R : (q, p) \mapsto (Q, P) = (Rq, Rp), \quad \psi_R : (\xi, \eta) \mapsto (\Xi, Y). \tag{5.1}$$

Let $K^{(r')}(q, p)$ be the PHO Hamiltonian with a HPP of degree-r', and $G^{(r')}(\xi, \eta)$ be its BG normal form up to degree-$2(r'-1)$, where the integer r' is not necessarily restricted to be odd. From $K^{(r')}(q, p)$ and $G^{(r')}(\xi, \eta)$, we induce the polynomials $\tilde{K}^{(r')}(Q, P)$ and $\tilde{G}^{(r')}(\Xi, Y)$ through the canonical transformations given by (5.1);

$$\tilde{K}^{(r')} \circ \phi_R = K^{(r')}, \quad \tilde{G}^{(r')} \circ \psi_R = G^{(r')}. \tag{5.2}$$

It is obvious from the nature of rotations that $\tilde{K}^{(r')}(Q,P)$ is the PHO Hamiltonian with another HPP of degree-r'. Using a more explicit expression [4] of the defining equation $G(\frac{\partial W}{\partial \eta}, \eta) = K(q, \frac{\partial W}{\partial q})$ in Table 1, we have the following:

Lemma 9 *$\tilde{G}^{(r')}$ is the BG normal form for $\tilde{K}^{(r')}$ up to degree-$2(r'-1)$.*

In other words, for the PHO Hamiltonian with every HPP, the BG normalization up to degree-$2(r'-1)$ and the rotational transformation commute. This is what the word 'equivariance' means.

We are now in a position to prove that the separability provides the sufficiency in Conjecture. We start with assuming that the PHO Hamiltonian $K^{(r)}(q,p)$ is separable with in a rotation generated by R, where the integer r will be assumed to be odd henceforce. Then $\tilde{K}^{(r)}(Q,P)$ induced through the rotation ϕ_R from $K^{(r)}(q,p)$ turns out to be the PHO Hamiltonian with a HPP of degree-r in the form,

$$\tilde{V}^{(r)}(Q) = \tilde{f}_0 Q_2^r + \tilde{f}_r Q_1^r, \tag{5.3}$$

so that we have

$$\tilde{G}^{(r)}(\Xi, Y) = \frac{1}{2}\sum_{j=1}^{2}(Y_j^2 + \Xi_j^2) + \mathcal{C}_{r-1,r-1}^{(r)}(Z_1\overline{Z}_1)^{r-1} + \mathcal{C}_{0,0}^{(r)}(Z_2\overline{Z}_2)^{r-1}, \tag{5.4}$$

from (4.8), where $Z_j = \Xi_j + iY_j$ $(j=1,2)$. It is easy to see from (4.7) that the PHO Hamiltonian,

$$\tilde{K}^{(2(r-1))}(Q,P) = \frac{1}{2}\sum_{j=1}^{2}(P_j^2 + Q_j^2) + \frac{4^{r-1}}{\begin{pmatrix} 2(r-1) \\ r-1 \end{pmatrix}}\left[\mathcal{C}_{0,0}^{(r)} Q_2^{2(r-1)} + \mathcal{C}_{r-1,r-1}^{(r)} Q_1^{2(r-1)}\right], \tag{5.5}$$

of degree-$2(r-1)$ shares $\tilde{G}^{(r)}(\Xi, Y)$ as the BG normal form up to degree-$2(r-1)$ with $\tilde{K}^{(r)}(Q,P)$. The equivariance for the BG normalization and the rotation yields that the PHO Hamiltonian $K^{(2(r-1))}(q,p)$ induced through (5.2) from $\tilde{K}^{(2(r-1))}(Q,P)$ of (5.5) shares the same BG normal form up to degree-$2(r-1)$ with the separable PHO Hamiltonian $K^{(r)}(q,p)$. To summarize, we have the following.

Theorem 10 *If the PHO Hamiltonian $K^{(r)}(q,p)$ with a HPP of degree-r is separable within a rotation, there exists the unique PHO Hamiltonian $K^{(2(r-1))}(q,p)$ with the HPP of degree-$2(r-1)$ which shares the same BG normal form up to degree-$2(r-1)$ with $K^{(r)}(q,p)$. The $K^{(2(r-1))}(q,p)$ is also separable by the same rotation as is used to the separation in $K^{(r)}(q,p)$.*

Remark Owing to Theorem 10, the PHO Hamiltonian of degree-$2(r-1)$ in Theorem 8 turns out to be separable.

Combining Theorems 8 and 10, we reach to the following conclusion:

Conclusion[5] *In the case of $r = 3, 5, 7$, Conjecture given in Sec.1 holds true.*

6. Concluding remarks

As is mentioned in Sec.4, computer algebra is indispensable for the proof, because the expression (4.8) is not explicit enough to draw (3.5) from (4.6) for general odd r. There seem to be two approaches to settle this difficulty in future. One is to think of a further use of computer algebra:

[4] See [1] and [2].

[5] The case $r = 3$ has been proved already in [1]

handle (4.6) symbolically on computer algebra under the constraints arising from the various relations among binomial coefficients and from the summation formulas. Another one is to study (4.8) from the combinatorial viewpoint. In both ways, computer algebra is expected to play a key role, especially the role of ‘experiment’.

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