Computational algebraic algorithms for the reliability of generalized k-out-of-n and related systems

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Identities and bounds for the reliability of coherent system are analysed and computed using the techniques of commutative algebra. Every multi-state coherent system has a monomial ideal associated with it and knowledge of its multigraded Betti numbers and/or multigraded Hilbert series provides exact reliability identities and good reliability bounds for the corresponding system. This arises from the fact that the Betti numbers appear in the minimal free resolution of the monomial ideal and from this it can be shown that the bounds obtained are tightest among all bounds based on resolutions. In particular they are typically tighter than the generalised Bonferroni-Fréchet inclusion-exclusion bounds which are associated with the Taylor resolution. There are different techniques to compute the multigraded Betti numbers of monomial ideals. Here Mayer-Vietoris trees are used, which allow a close analysis of the ideals from a homological point of view and give rise to a natural algorithm.

An important family of coherent systems which have been the object of considerable attention in the reliability literature are **k-out-of-n systems**. They have also been at the centre of the study of scan statistics, used, for example, in gene association studies. The techniques are applied to the analysis of some of the most relevant systems in this family. The k-out-of-n systems, consecutive k-out-of-n systems, weighted k-out-of-n and multistate k-out-of-n systems are used to benchmark the methods.

The algorithms to compute the multigraded Betti numbers lead to recurrence relationships, which can be solved to give special generating functions and explicit formulas for the numbers and for the identities and bounds. From there it is possible to derive asymptotic formulae for exact reliability and bounds under models such as independence as, for example, $n \to \infty$. Some of these results are known from probability theory but many are new and compare well with bounds derived by other methods. This demonstrates that the algebra of monomial ideals can have significant application in the probability of certain discrete systems and points to a new algebra/probability interface.

When computing the reliability of coherent systems using the commutative algebra approach, one has to compute either the Betti numbers of the corresponding ideal or its multigraded Hilbert series. The ideals corresponding to actual systems appearing in applications, typically have a large number of

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variables, which makes many problems difficult to handle for computer algebra systems. Our algorithm, based on Mayer-Vietoris trees, and implemented in the `C++` library `CoCoALib` gives good results, in particular when the number of variables grows, which makes it suitable for this kind of application. Our algorithms provide the form of the Hilbert series that is needed for the analysis of the reliability of systems in comparatively high number of variables.

Tables and diagrams are given to show the exact identities and bounds, the comparison with their asymptotic formulae and, where they exist, with results from the literature. Actual applications of several coherent systems to problems on networks, project management, etc.. are stressed that show that our analysis and algorithms yield an efficient way to evaluate their reliability. For a range of other problems, when there are no combinatorial formulae or asymptotics results the computer algorithms still give fast accurate solutions for relatively large systems.

The consecutive k-out-of-n systems lead to one type of *scan statistic* for use in genome wide associations studies where it is computationally infeasible to produce statistical tests which need to store and use the gene expression for a very long sequence of length n. By scanning and using results for windows of length k in a sequential manner, great savings can be made. The computational methods described here may be used to develop new approaches for statistical testing and evaluating false discovery rates (FDR) in this environment.

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