Parametric characterization of hypercircles

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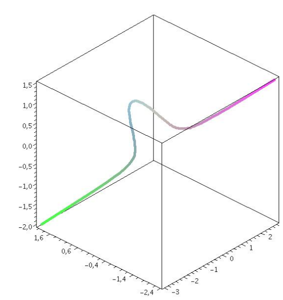
We can think of the real plane as the field of complex numbers $\mathbb{C}$, an algebraic extension of the reals $\mathbb{R}$ of degree 2. Analogously, we can consider a characteristic zero base field $\mathbb{K}$ and an algebraic extension of degree n, $\mathbb{K}(\alpha) = \mathbb{K}[\alpha]$, i.e. expressions given by $\mathbb{K}$-polynomials in α up to degree $n-1$ (which turns to include as well all quotients of such expressions, exactly in the same way as a quotient of two complex numbers –a $\mathbb{R}$-polynomial of degree 1 in the letter i– is again a complex number). Let us identify $\mathbb{K}(\alpha)$ as the vector space $\mathbb{K}^n$, via the choice of a suitable base, such as the one given by the powers of α.

Then, recall that a real circle can be defined as the image (in the real plane, suitably identified with the complex numbers) of the real axis under a Moebius transformation (of the kind $\frac{at+b}{ct+d}$, with $a, b, c, d \in \mathbb{C}$) in the complex field. Likewise, and roughly speaking, a hypercircle (i.e. a *non-standard circle*) can be defined as the curve in $\mathbb{K}^n$ that is the image of "the $\mathbb{K}$-axis" under the transformation $\frac{at+b}{ct+d} : \mathbb{K}(\alpha) \to \mathbb{K}(\alpha)$. They have been introduced in [1] and studied in detail in [2].

For example, if we take $\mathbb{K} = \mathbb{Q}$ and α such that $\alpha^3 + 2 = 0$, and we consider the map $\Phi = \dfrac{t+\alpha}{t-\alpha}$, we obtain the hypercircle in $\mathbb{Q}^3$ parametrized (i.e. obtained as the image of a mapping from $\mathbb{Q} \longrightarrow \mathbb{Q}^3$ defined) by $[\dfrac{t^3-2}{2+t^3}, \dfrac{2t^2}{2+t^3}, \dfrac{2t}{2+t^3}]$, with plot as follows



The study of these hypercircles is fascinating and opens the door to stating many different questions. For instance, circles, through classical Moebius transformations, are related to conformal (=angle perserving) geometry and to complex holomorphic (i.e. functions of one complex variable are complex-differentiable at every point) functions. Which analogous notions of angle-perserving and holomorphic functions could be defined through the general

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framework that allows the definition of hypercircles? We think that the possibility of connecting that part of Mathematics to Computer Algebra is, again, highly *non-standard*.

In this direction, we will focus in our communication on a humble and basic problem. Given an algebraic extension of degree n, $\mathbb{K}(\alpha)$, and a parametric curve in $\mathbb{K}^n$, when is it a hypercircle? And, if so, we want to algorithmically identify the transformation $\frac{at+b}{ct+d} : \mathbb{K}(\alpha) \to \mathbb{K}(\alpha)$ yielding the hypercircle. Notice that, for the classical case of standard circles, the problem is to determine if a plane curve, given by a parametrization (perhaps with complex coefficients), is a circle and, in the affirmative case, to find its geometric elements, since they determine the Moebius transformation.

We will present a complete and algorithmic solution to both questions for hypercircles and will briefly comment on the following application of this (on the other hand quite natural) problem. Assume a planar or spatial rational curve $\mathcal{C}$ is given by a parametrization over $\mathbb{K}(\alpha)$. Then, we want to obtain, whenever possible, a simpler parametrization over $\mathbb{K}$ of the same curve $\mathcal{C}$. In [1] it is shown that this problem is reduced to determining that a certain curve is a hypercircle. Moreover, if we have a $\mathbb{K}$ parametrization of this hypercircle, a $\mathbb{K}$ parametrization of the original curve is then achieved by a simple substitution. This may-be hypercircle is found manipulating algorithmically the parametrization of the originally given curve, by a method analogous to Weil's descent (see [3] for a detailed description of this procedure).

So, the communication we propose here contributes to closing the solution to this simplification problem, since it allows to algorithmically decide if a given curve is a hypercircle and to parametrize it over $\mathbb{K}$, which is trivial once the corresponding "Moebius" transformation is known.

References

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