A New Criterion for the Existence of Real Zeros of Polynomial Systems

Jin-San Cheng
KLMM, Institute of Systems Science, AMSS, CAS (China)

jcheng@amss.ac.cn

Abstract

In both theory and practice, dealing with multiple zeros and clusters of zeroes or components with positive dimension of multivariate polynomial system is a challenging problem. We give a theoretical result to the problem.

Denote $\Sigma = \{f_1, \ldots, f_m\} \subset \mathbb{R}[x_1, \ldots, x_n]$, $f = \sum_{i=1}^m f_i^2$, $\mathfrak{S}_f = \{c \in \mathbb{C} : f - c \text{ is singular}\}$, $\Sigma_r = \{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f - r\}$, where $\mathbb{R}, \mathbb{C}$ are fields of real and complex numbers respectively.

Definition 1 *We say a point P* **is attracted to a component** Q **of** $V_{\mathbb{R}}(\Sigma_{\bar{r}})(\bar{r} \in \mathfrak{S}_f)$ *when $r \to \bar{r}$ (simply for P is attracted to a component Q without misunderstanding) if there exists a path C from P to a point $P' \in Q$ such that the value of $r = f$ at point $R \in C$ decreases to $\bar{r}$ when R moves from P to P'.*

We give a new criterion for numerically deciding whether a real point $P \in \mathbb{R}^n$ is attracted to a real zero (regular, multiple or a point on a component with positive dimension) of a polynomial system.

Theorem 1 *Let $\Sigma = \{f_1, \ldots, f_m\} \subset \mathbb{R}[x_1, \ldots, x_n]$. Then there exists a real number $r_0 > 0$ such that for any $P \in \mathbb{R}^n$, if*

$$f(P) = \sum_{i=1}^m f_i^2(P) < r_0,$$

then P is attracted to a component of $V_{\mathbb{R}}(\Sigma) \neq \emptyset$.

Given two points $P, Q \in \mathbb{R}^n$ that are attracted to some real zeros of Σ, we give a criterion to judge whether both P, Q are attracted to the same real connected component of Σ.

Theorem 2 *Let $\Sigma = \{f_1, \ldots, f_m\} \subset \mathbb{R}[x_1, \ldots, x_n]$. Let $P_1, P_2 \in \mathbb{R}^n$ be two points and $f(P_i) < r_0 (i = 1, 2)$, where $f = \sum_{i=1}^m f_i^2$. They are both attracted to the same component of $V_{\mathbb{R}}(\Sigma)$ if and only if there exists a path $C(P_1P_2)$ such that for any point P on $C(P_1P_2)$,*

$$f(P) \leq \max\{f(P_1), f(P_2)\}.$$

Keywords
polynomial system, real zeros, certified numerical solving