Gröbner-Shirshov Basis and Reduced Words for Affine Weyl Group $\widetilde{A_n}$

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Abstract

Gröbner and Gröbner-Shirshov bases theories are generating increasing interest because of its usefulness in providing computational tools and in giving algebraical structures which are applicable to a wide range of problems in mathematics, science, engineering, and computer science. In particular, Gröbner and Gröbner-Shirshov bases theories are powerful tools to deal with the normal form, word problem, embedding problem, extensions of algebras, Hilbert series, etc. The true significance of Gröbner-Shirshov bases is the fact that they can be computed.

Gröbner-Shirshov basis and normal form of the elements were already found for the Coxeter groups of type A_n, B_n and D_n in [1]. They also proposed a conjecture for the general form of Gröbner-Shirshov bases for all Coxeter groups. In [2], the example was given to show that the conjecture is not true in general. The Gröbner-Shirshov bases of the other finite Coxeter groups are given in [3] and [4]. This paper is the first example of finding Gröbner-Shirshov bases for an infinite Coxeter group, defined by generators and defining relations.

The main purpose of this paper is to find a Gröbner-Shirshov basis and classify all reduced words for the affine Weyl group $\widetilde{A}_n$. The strategy for solving the problem is as follows.

Let R be the set of polynomials of the defining relations of $\widetilde{A}_n$. Using Buchberger-Shirshov algorithm, we obtain a new set R' of polynomials including R. Then, by using the algorithm of elimination of leading words with respect to the polynomials in R', all the words in the group $\widetilde{A}_n$ are reduced to the explicit classes of words. After that, using combinatorial techniques, we compute the number of all reduced words with respect to these classes by means of a generating function. This generating function turns out to be same with the well known Poincaré polynomial of the affine Weyl group $\widetilde{A}_n$. Therefore, by the Composition-Diamond Lemma the functions in R' form a Gröbner-Shirshov basis for the affine Weyl group $\widetilde{A}_n$. Furthermore, one can easily see that this basis is in fact a reduced Gröbner-Shirshov basis.

Keywords

Affine Weyl Groups, Gröbner-Shirshov Basis, Composition-Diamond Lemma, q-binomials

References

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