Padovan-like sequences and Bell polynomials

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Abstract

We study a class of Padovan-like sequences that can be generated using special matrices of the third order. We show that terms of any sequence of this class can be expressed via Bell polynomials and their derivatives that use as arguments terms of another such sequence with smaller indexes. CAS *Mathematica* is used for cumbersome calculations and hypothesis testing.

Keywords
Padovan sequence, Fibonacci numbers, Bell polynomials, integer sequences

1 Introduction

Integer sequences appear in many branches of science. One famous example is Fibonacci numbers that have been known for more than two thousand years and find applications in mathematics, biology, economics, computer science. Padovan numbers are much younger - they were introduced only recently [1]. Below, we will study Padovan-like sequences that can be generated using special matrices of the third order. We will find expressions for terms of one sequence in terms of another sequence via Bell polynomials. CAS *Mathematica* was used for cumbersome calculations and hypothesis testing.

Let

$$A_\alpha = \begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and let $\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$ for a time denote the first column of A_α^n, $n \geq 0$. Then

$$u_{n+1} = \alpha v_n + w_n, \quad v_{n+1} = u_n, \quad w_{n+1} = v_n. \tag{1}$$

We have

$$u_{n+1} = \alpha u_{n-1} + u_{n-2}, \quad u_0 = 1, \quad u_1 = 0, \quad u_2 = \alpha, \quad (u_{-1} = 0,) \tag{2}$$

$$A_\alpha^n = \begin{pmatrix} u_n & u_{n+1} & u_{n-1} \\ u_{n-1} & u_n & u_{n-2} \\ u_{n-2} & u_{n-1} & u_{n-3} \end{pmatrix}. \tag{3}$$

Some examples of sequences generated by the matrix A_α with references to the On-Line Encyclopedia of Integer Sequences (OEIS, http://oeis.org/) are given in Table 1.

Table 1: Examples of sequences generated by the matrix A_α.

α	OEIS reference	First terms	Comment
$\alpha=1$	A000931	1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,	Padovan sequence : $u_n = p_n$
$\alpha=2$	A008346	1,0,2,1,4,4,9,12,22,33,56,88,145,	$u_n = f_n = Fibonacci(n) + (-1)^n$
$\alpha=3$	A052931	1,0,3,1,9,6,28,27,90,109,297,417,	
$\alpha=0$	A079978	1,0,0,1,0,0,1,0,0,1,0,0,1,0,0, 1,	$u_n = t_n = \begin{cases} 1, & n \equiv 0 (mod\, 3) \\ 0, & n \equiv 1,2 (mod\, 3) \end{cases}$ $= \binom{\frac{n}{3}}{0}$
$\alpha=-1$	A077961	1,0,-1,1,1,-2,0,3,-2,-3,5,1,-8,4,9,	
$\alpha=-2$	A077965	1,0,-2,1,4,-4,-7,12,10,-31,-8,72,-15,-152,102	

2 Main Result

Now let $\beta \in Z$, and let v_n denote terms of the sequence corresponding to the powers of the matrix A_β, and let w_n denote terms of the sequence corresponding to the powers of the matrix $A_{\alpha+\beta}$:

$$A^n_{\alpha+\beta} = \begin{pmatrix} w_n & w_{n+1} & w_{n-1} \\ w_{n-1} & w_n & w_{n-2} \\ w_{n-2} & w_{n-1} & w_{n-3} \end{pmatrix} = \begin{pmatrix} 0 & \alpha+\beta & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = (A_\alpha + \beta e)^n = (A_\beta + \alpha e)^n,$$

where $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Terms of the sequence generated by $A_{\alpha+\beta}$ can be expressed in terms of the sequence generated by A_α as follows:

$$\begin{aligned} w_n &= \sum_{\varepsilon=0}^{1} \sum_{s=1}^{\left[\frac{n+\varepsilon}{2}\right]} \frac{(s-\varepsilon)! \cdot \beta^{s-\varepsilon}}{(n-s+\varepsilon)!} \times \\ &\quad \times \mathcal{D}^\varepsilon (B_{n-s+\varepsilon,s}(1! \cdot u_{k_1-1}, 2! \cdot u_{k_2-1}, \ldots, (n-2s+1+\varepsilon)! \cdot u_{k_{n-2s+1+\varepsilon}-1})), \end{aligned} \tag{4}$$

where k_i run over all partitions of $n-s+\varepsilon$ into s parts, $\mathcal{D}^0 = id$ - identity operator, $\mathcal{D}^1 = \mathcal{D} = \sum_{r=1}^{(\infty)} x_{r+1} \frac{\partial}{\partial x_r}$, and

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \ldots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}} \tag{5}$$

are partial Bell polynomials, and summing is done for all sets $j_1, j_2, \ldots, j_{n-k+1}$ of non-negative integers such that $j_1 + j_2 + \ldots + j_{n-k+1} = k$ and $1 \cdot j_1 + 2 \cdot j_2 + 3 \cdot j_3 + \ldots + (n-k+1) \cdot j_{n-k+1} = n$ (see [2].)

3 Examples

3.1 Example 1

For $\alpha = \beta = 1$ formula (4) gives a relation between terms of Fibonacci and Padovan sequences:

$$\begin{aligned} f_n &= Fibonacci(n) + (-1)^n = \\ &= \sum_{\varepsilon=0}^{1} \sum_{s=1}^{\left[\frac{n+\varepsilon}{2}\right]} \frac{(s-\varepsilon)!}{(n-s+\varepsilon)!} \mathcal{D}^{\varepsilon}(B_{n-s+\varepsilon,s}(1! \cdot p_{k_1-1}, 2! \cdot p_{k_2-1}, \ldots, (n-2s+1+\varepsilon)! \cdot p_{k_{n-2s+1+\varepsilon}-1})), \end{aligned} \tag{6}$$

and thus for $n \geq 1$ the $n-$th term of the sequence $Fibonacci(n) + (-1)^n$ is expressed as sums of products of the n first terms of the Padovan sequence.

3.2 Example 2

For $\alpha = 2$ and $\beta = -2$ formula (4) gives a relation between terms of sequences $f_n = Fibonacci(n) + (-1)^n$ and $t_n = \begin{cases} 1, & n \equiv 0 \ (mod\ 3) \\ 0, & n \equiv 1, 2 \ (mod\ 3) \end{cases}$, $n \geq 1$:

$$\sum_{\varepsilon=0}^{1} \sum_{s=1}^{\left[\frac{n+\varepsilon}{2}\right]} \frac{(s-\varepsilon)!(-2)^{s-\varepsilon}}{(n-s+\varepsilon)!} \mathcal{D}^{\varepsilon}(B_{n-s+\varepsilon,s}(1! \cdot f_{k_1-1}, 2! \cdot f_{k_2-1}, \ldots, (n-2s+1+\varepsilon)! \cdot f_{k_{n-2s+1+\varepsilon}-1})) = \\ = \begin{cases} 1, & n \equiv 0 \ (mod\ 3) \\ 0, & n \equiv 1, 2 \ (mod\ 3) \end{cases} \tag{7}$$

3.3 Example 3

In particular, the terms of every Padovan-like sequence with $\alpha \neq 0$ can be expressed in the same way via the terms of the "simplest generator" of the class, namely the sequence $1, 0, 0, 1, 0, 0, 1, 0, 0, ...$ generated by $\alpha = 0$ (OEIS-number A079978, see Table 1). This statement is a far-reaching generalization of the result announced by J. Vladetta in [3].

References

[1] Richard Padovan, Dom Hans Van Der Laan and the Plastic Number, pp. 181-193 in Nexus IV: *Architecture and Mathematics*, eds. Kim Williams and Jose Francisco Rodrigues Fucecchio (Florence): Kim Williams Books, 2002.

[2] http://en.wikipedia.org/wiki/Bell_polynomials

[3] Vladetta.J , in "Sloane's A000931 : Padovan sequence", The On-Line Encyclopedia of Integer Sequences. OEIS Foundation.