Lagrange's Theorem of 1767(1769) for Computing Bounds on the Values of the Positive Roots of Polynomials

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Abstract

In our previous attempt to develop the best bound on the values of the positives roots of polynomials we totally missed — for reasons explained in the talk — Lagrange's[1] theorem of 1767(1769).

In this paper we present this almost forgotten theorem by Lagrange, along with its interesting history and a short proof of it dating back to 1842. Since the bound obtained by Lagrange's theorem is of linear complexity, in the sequel it is called "Lagrange Linear", or `LL` for short.

Despite its average good performance, `LL` is endowed with the weaknesses inherent in *all* bounds with linear complexity and, therefore, the values obtained by it can be much bigger than those obtained by our own bound "Local Max Quadratic", or `LMQ` for short.

To level the playing field, we incorporate Lagrange's theorem into our `LMQ` and we present the new bound "Lagrange Quadratic", or `LQ` for short, the quadratic complexity version of `LL`. It turns out that `LQ` is one of the most efficient bounds available since, at best, the values obtained by it are half of those obtained by `LMQ`.

Empirical results indicate that when `LQ` replaces `LMQ` in the Vincent-Akritas-Strzeboński Continued Fractions (`VAS-CF`) real root isolation method, the latter becomes measurably slower for some classes of polynomials.

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[1] Italian mathematician Joseph-Louis Lagrange, born Giuseppe Lodovico Lagrangia, (25 January 1736 - 10 April 1813).