Familiarizing students with definition of Lebesgue integral - examples of calculation directly from its definition using Mathematica

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> *"Young man, in mathematics you don't understand things. You just get used to them"*
>
> John von Neumann

In popular books of calculus, for example [1, 2], we can find many examples of Riemann integral calculated directly from its definition. The aim of these examples is to familiarize students with the definition of Riemann integral. But we cannot find analogical examples for Lebesgue integral. In this article, with similar aim but for Lebesgue integral definition, we present the following examples of calculation directly from its definition: $\int_0^1 x^2\,\mathrm{d}m(x)$, $\int_0^1 x^k\,\mathrm{d}m(x)$, $\int_0^{\pi/2} \sin x\,\mathrm{d}m(x)$, $\int_a^b \exp(x)\,\mathrm{d}m(x)$, $\int_0^{\pi} \ln(1-2r\cos x+r^2)\,\mathrm{d}m(x)$, where $\mathrm{d}m(x)$ denotes the Lebesgue measure on the real line. We calculate sums, limits and plot graphs of needed simple functions using Mathematica. The two following definitions of Lebesgue integral are used in this article:

Let $(\mathbb{R},\mathfrak{M},m)$ be measure space, where $\mathfrak{M}$ is $\sigma-$ algebra of Lebesgue measurable subsets in $\mathbb{R}$, and m- Lebesgue measure on $\mathbb{R}$.

Let $f:\mathbb{R}\to\mathbb{R}$ be measurable nonnegative function (we've omitted the definition of Lebesgue integral for simple real measurable functions).

Definition 1. (See [3, 5, 6, 7, 8])

$$\int f\,\mathrm{d}m(x) = \sup\left\{\int s\,\mathrm{d}m(x) : 0 \le s \le f, s \text{ simple measurable function}\right\}. \quad (1)$$

Definition 2. (See [4, 9, 10]) Let s_n be nondecreasing sequence of nonnegative simple measurable functions such that $\lim\limits_{n\to\infty} s_n(x) = f(x)$ for every $x\in\mathbb{R}$. Then:

$$\int f\,\mathrm{d}m(x) = \lim_{n\to\infty}\int s_n\,\mathrm{d}m(x). \quad (2)$$

Example. Let consider the function: $f(x) = \sin x, x \in [0, \pi/2)$. For the rest of this example we will restrict our consideration to $x \in [0, \pi/2)$.

We will calculate $\displaystyle\int_0^{\pi/2} \sin x \, \mathrm{d}m(x)$ applying directly definition 1.

Consider

$\underline{s}_n(x) = \displaystyle\sum_{k=0}^{2^n-1} \sin\left(\frac{k}{2^{n+1}}\pi\right)\chi_{\left[\frac{k}{2^{n+1}}\pi, \frac{k+1}{2^{n+1}}\pi\right)}(x)$, for $x \in [0, \pi/2)$, $n = 1, 2, \ldots$

and $\bar{s}_n(x) = \bar{s}_n = \displaystyle\sum_{k=1}^{2^n} \sin\left(\frac{k}{2^{n+1}}\pi\right)\chi_{\left[\frac{k-1}{2^{n+1}}\pi, \frac{k}{2^{n+1}}\pi\right)}(x)$, for $x \in [0, \pi/2)$, $n = 1, 2, \ldots$.

Using Wolfram Mathematica we get the following Figures 1, 2:

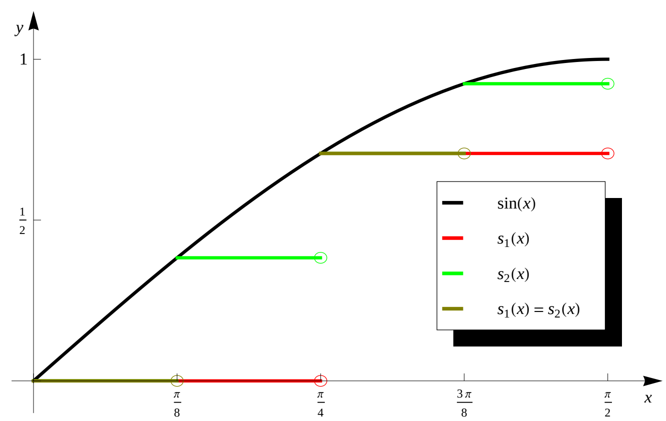

Figure 1: Graphs of functions $f, \underline{s}_1, \underline{s}_2$. We can see that $\underline{s}_1(x) \leq \underline{s}_2(x)$ for $x \in [0, \pi/2)$.

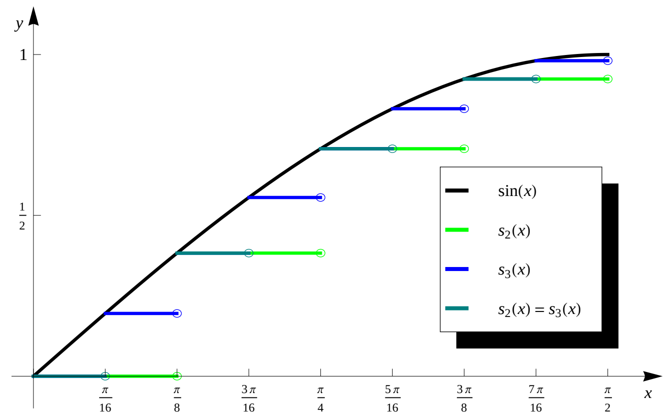

Figure 2: Graphs of functions $f, \underline{s}_2, \underline{s}_3$. We can see that $\underline{s}_2(x) \leq \underline{s}_3(x)$ for $x \in [0, \pi/2)$.

It is clear that $\underline{s}_n$, $\bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, \pi/2)$ for all $n = 1, 2, \ldots$.

Using Wolfram Mathematica we get:

Listing 1: Mathematica code:

```
In[1]=Sum[Sin[Pi k/2^(n+1)], {k, 0, 2^n-1}] Pi/2^(n+1)//Simplify
Out[1]=2^(-2-n)Pi(-1+Cot[2^(-2-n)Pi])

In[2]=Limit[%,n->Infinity]
Out[2]=1
```

So

$$\underline{a}_n = \int \underline{s}_n \,\mathrm{d}m)x) = \sum_{k=0}^{2^n-1} \sin \frac{k\pi}{2^{n+1}} \cdot \frac{1}{2^{n+1}} \pi = 2^{-2-n} \pi(-1 + \cot(2^{-2-n}\pi)) \to 1 \quad (3)$$

Similarly

Listing 2: Mathematica code:

```
In[3]=Sum[Sin[Pi k/2^(n+1)], {k, 1, 2^n}] Pi/2^(n+1)//Simplify
Out[3]=2^(-2-n)Pi(1+Cot[2^(-2-n)Pi])
```

```
In[4]=Limit[%,n->Infinity]
Out[4]=1
```

So

$$\bar{a}_n = \int \bar{s}_n \, \mathrm{d}m(x) = \sum_{k=1}^{2^n} \sin \frac{k\pi}{2^{n+1}} \cdot \frac{1}{2^{n+1}} \pi = 2^{-2-n} \pi (1 + \cot(2^{-2-n}\pi)) \to 1 \tag{4}$$

Of course we could use the following formulae: $\sum_{k=1}^{n} \sin(kx) = \dfrac{\sin \frac{n+1}{2} x \sin \frac{n}{2} x}{\sin \frac{x}{2}}$ and $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ instead of the code in Listings 1 and 2 to get the results in formulae (3) and (4).

Using formulae (3) and (4), basic properties of least upper, greatest lower bounds and of Lebesgue integral of simple measurable functions we will prove in our talk that:

$$\sup \left\{ \int s \, \mathrm{d}m(x) : 0 \le s \le f, s \text{ simple measurable function} \right\} \ge 1 \tag{5}$$

and

$$\sup \left\{ \int s \, \mathrm{d}m(x) : 0 \le s \le f, s \text{ simple measurable function} \right\} \le 1. \tag{6}$$

Inequalities (5) and (6) give

$$\sup \left\{ \int s \, \mathrm{d}m(x) : 0 \le s \le f, s \text{ simple measurable function} \right\} = 1,$$

which means that $\int f \, \mathrm{d}m(x) = \int \sin x \, \mathrm{d}m(x) = 1$.

Let calculate $\int \sin x \, \mathrm{d}m(x)$ applying directly definition 2.

We can see that $\underline{s}_n(x) \le \underline{s}_{n+1}(x)$ for $x \in [0, \pi/2)$ and for all $n = 1, 2, \ldots$. In Figures 1 and 2 we can see that $\underline{s}_1(x) \le \underline{s}_2(x)$ and $\underline{s}_2(x) \le \underline{s}_3(x)$ for $x \in [0, \pi/2)$. W can also see that $\lim_{n\to\infty} \underline{s}_n(x) = \sin(x)$ for all $x \in [0, \pi/2)$. So $\underline{s}_n$ is nondecreasing sequence of nonnegative simple measurable functions and $\underline{s}_n$ converges pointwise to f.

So by formula (3) and directly by definition 2 we get $\int \sin x \, \mathrm{d}m(x) = \int f \, \mathrm{d}m(x) = \lim_{n\to\infty} \int \underline{s}_n \, \mathrm{d}m(x) = \lim_{n\to\infty} \underline{a}_n = 1$.

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