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Detecting "true on components" statements in automated reasoning in geometry*

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We investigate and generalize to an extended framework the notion of *true on components* introduced by Zhou, Wang and Sun in their paper [7]. A new, simple criterion is presented for a statement to be simultaneously not generally true and not generally false (i.e. true on components), and its performance is exemplified through the implementation of this test in the dynamic geometry program *GeoGebra*.

The algebraic geometry approach to automated reasoning in geometry proceeds by translating a geometric statement $\{H \Rightarrow T\}$ into polynomial expressions, after adopting a coordinate system. Then, the geometric instances verifying the hypotheses can be represented as the solution of a system of polynomial equations $V(H) = \{h_1 = 0, \ldots, h_r = 0\}$ (*hypotheses variety*) they are represented algebraically by the ideal (of hypotheses) $H = \langle h_1 = 0, \ldots, h_r = 0\rangle$ generated by such polynomials. Analogously, the thesis is represented as the solution of a polynomial $V(T) = \{f = 0\}$, describing the hypotheses (resp. the thesis) variety.

Thus, when $V(H) \subseteq V(T)$ we can say that the theorem is *always true*. But this fact rarely happens, even for well established theorems, because the algebraic translation of the geometric construction described by the hypotheses usually forgets explicitly excluding some degenerate cases, cf. [4].

Thus, a delicate, but more useful, approach for automated reasoning consists in exhibiting, first, a collection of independent variables modulo H, so that no polynomial relation among them holds over the whole $V(H)$ (*independent variables modulo* H). Now, the irreducible components of $V(H)$ where these variables do remain independent are assumed to describe *non-degenerate* instances.

Accordingly, a statement is called *generally true* if the thesis holds, at least, over all the non-degenerate components. On the other hand, if over each non-degenerate component the thesis does not identically vanish, the statement is labeled as *generally false*. Remark that this last includes the *always false* case, where the thesis does not hold at all. A more detailed description of this quite established terminology (with small variants) can be consulted, for instance, at [6], [3] or [7]. It follows from the definition that to be generally true and to be generally false are incompatible. However—and this is the object of interest in this paper—there are statements which happen to be, simultaneously, not generally true and not generally false, i.e. statements that are called [7] *true on components*, in some specific sense we will describe in detail below.

Let us first start analyzing a simple example. Consider points $A(0,0)$, $B(2,0)$ in the plane and construct circles $c = (x-0)^2+(y-0)^2-3$ and $d = (x-2)^2+(y-0)^2-3$, i.e. circle c is centered at A and circle d is centered at B and both have the same radius $r = \sqrt{3}$. Finally, we consider the two points of intersection of these circles, namely, $E(u,v)$ and $F(m,n)$. Thus, the hypotheses ideal is $\left\langle u^2 + v^2 - 3, (u-2)^2 + v^2 - 3, m^2 + n^2 - 3, (m-2)^2 + n^2 - 3 \right\rangle$.

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The thesis states the parallelism of the lines AE and BF, that is, the vanishing of the polynomial $u \cdot n - v \cdot (m-2)$. The ideal of hypotheses is clearly zero-dimensional, so there are no independent variables, nor degenerate components. Its primary components, over the rationals, are

$$\langle v-n, (m-2)^2+n^2-3, (u-2)^2+v^2-3, m^2+n^2-3, u^2+v^2-3 \rangle$$
and
$$\langle v+n, (m-2)^2+n^2-3, (u-2)^2+v^2-3, m^2+n^2-3, u^2+v^2-3 \rangle.$$

It easy to check that the thesis is false over the first one and true over the second. This a clear, simple example of a neither true nor false, i.e. of a *true on components*, statement arising in an elementary geometry context (see other, less artificial examples in [6, 1]).

Obviously, since the idea of *true on components* is based on the concepts of degeneracy and of irreducible component, it follows that both the choice of the field over which the prime decomposition is performed (for example, the ideal H of the previous example has four components instead, if $\mathbb{Q}(\sqrt{2})$ is considered as base field) and the choice of the independent variables modulo H are essential.

About this last issue we would like to remark that when dealing with geometric statements it seems logical to take as independent variables the coordinates of the free points in the geometric construction we are dealing with; and we expect that its cardinality is the dimension of the hypotheses ideal. In most cases this "intuitively" maximal set of independent variables is maximum-size, but there are examples in which the coordinates of the free points in the geometric construction do not provide a maximum-size set of independent variables. See, for instance, Example 7 in [4], concerning Euler's formula regarding the radii of the inner and outer circles of a triangle with vertices $(-1,0), (1,0), (u[1], u[2])$. Here the dimension of the hypotheses variety is expected to be 2 (referring to the two coordinates of the only free vertex of the triangle), but applying the algebraic definition of independence it turns out to be three..., unless it is explicitly required, and added as a new hypothesis, that $(u[1], u[2])$ does not lie in the x-axis! This is a quite common problem—related, as mentioned above, to the difficult *a priori* control and detail of all geometric degeneracies—and is already considered in the basic reference of [2].

The aim of this talk is to justify the specific interest of "true on components" statements in the context of automated reasoning in geometry, pointing out the subtle, involved, issues deriving from the quirky algebraic behavior described in some of the examples above, as well as exhibiting a new, more general and simpler way than in [7], of testing if a statement is true on components, by just detecting if a pair of elimination ideals is zero or not. This test has been implemented in the dynamic geometry software GeoGebra and some illustrative examples can be found in `https://www.geogebra.org/m/zpDq7taB`.

This extended abstract is based on a recent work by the authors [5].

Keywords: geometry theorem proving and discovery, elementary geometry, Gröbner basis, elimination, true on components

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