

SPECIAL SESSIONS

Applications of Computer Algebra - ACA2018



June 18–22, 2018

Santiago de Compostela, Spain

S3

Computer Algebra in Education

Tuesday

Tue 19th, 10:30 - 11:00, Aula 8 – Penka Georgieva:

About the Bulgarian experience in organizing National Student Olympiad in Computer Mathematics

Tue 19th, 11:30 - 12:00, Aula 8 – Karsten Schmidt:

Student Attitudes toward Technology Use in Math Education

Tue 19th, 12:00 - 12:30, Aula 8 – Elena Varbanova:

Technology enhanced e-assessments in Calculus courses with application of CAS

Tue 19th, 12:30 - 13:00, Aula 8 – Rein Prank:

Analyzing the “Calculator Effect” of Different Kinds of Software for School Arithmetics and Algebra

Tue 19th, 13:00 - 13:30, Aula 8 – Włodzimierz Wojas – Jan Krupa:

Dynamic visualizations for network flow optimizations problems with Mathematica

Tue 19th, 15:30 - 16:00, Aula 8 – Hanan Smidi:

Using TI-Nspire for the financial education of future engineers

Tue 19th, 16:00 - 16:30, Aula 8 – David G. Zeitoun:

Accurate plotting in 3D: how to choose the mesh

Tue 19th, 16:30 - 17:00, Aula 8 – Anouk Bergeron-Brelek:

Addressing discrete mathematics problems in the classroom

Tue 19th, 17:30 - 18:00, Aula 8 – Louis-Xavier Proulx:

Introducing parametric curves with CAS

Tue 19th, 18:00 - 18:30, Aula 8 – José Luis Galán-García:
New rules for improving Cas capabilities when computing improper integrals. Applications in Math Education

Tue 19th, 18:30 - 19:00, Aula 8 – José Luis Galán-García:
Teaching Partial Differential Equations with CAS

Wednesday

Wed 20th, 10:00 - 10:30, Aula 8 – Eugenio-Roanes Lozano:
Do we take advantage of ICT when teaching maths at primary and secondary education levels? Do we teach maths as we should?

Wed 20th, 10:30 - 11:00, Aula 8 – Włodzimierz Wojas – Jan Krupa:
Visualizations of the nondominated set and the efficient set in multicriteria optimization problems using Mathematica

Wed 20th, 11:30 - 12:00, Aula 8 – Gilbert Labelle:
Analyzing discrete suspended chains using computer algebra

Wed 20th, 12:00 - 12:30, Aula 8 – Thierry Dana-Picard:
Fractals and tessellations: from K 's to cosmology

Wed 20th, 12:30 - 13:00, Aula 8 – Michel Beaudin:
Periodic and Nontrivial Periodic Input in Linear ODEs (Part I)

Wed 20th, 13:00 - 13:30, Aula 8 – Michel Beaudin:
Periodic and Nontrivial Periodic Input in Linear ODEs (Part II)

Thursday

Thu 21st, 10:00 - 10:30, Aula 8 – Anatoli Kouropatov:
Consolidation of abstract knowledge in the process of confronting errors using digital tools: The case of the inflection point

Thu 21st, 10:30 - 11:00, Aula 8 – Leili Rafiee Sevyeri:
The Runge Example for Interpolation and Wilkinson's Examples for Rootfinding

Thu 21st, 11:30 - 12:00, Aula 8 – Michael Xue:
A non-iterative method for solving nonlinear equations

Thu 21st, 12:00 - 12:30, Aula 8 – David J. Jeffrey:
What is the integral of x^n ?

Thu 21st, 12:30 - 13:00, Aula 8 – Benjamin Jurell:
CAS in Teaching Basics of Stereoscopy

Thu 21st, 16:30 - 17:00, Aula 8 – Włodzimierz Wojas – Jan Krupa:
*Familiarizing students with definition of Lebesgue measure using Mathematica –
some examples of calculation directly from its definition*

Organizers

Michel Beudin:

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Aim and scope

Education has become one of the fastest growing application areas for computers in general and computer algebra in particular. Computer Algebra Systems (CAS) make for powerful teaching and learning tools within mathematics, physics, chemistry, biology, economics, etc. Among them are:

- the commercial “heavy weights” such as Casio ClassPad 330, Derive, Magma, Maple, Mathematica, MuPAD, TI NSpire CAS, and
- the free software/open source systems such as Axiom, Euler, Fermat, wxMaxima, Reduce, and the rising stars such as GeoGebra, Sage, SymPy and Xcas (the swiss knife for mathematics).

The goal of this session is to exchange ideas, discuss classroom experiences, and to explore significant issues relating to CAS tools/use within education. Subjects of interest for this session will include new CAS-based teaching/learning strategies, curriculum changes, new support materials, assessment practices from all scientific fields, and experiences of joint use of applied mathematics and CAS.

We emphasize that all levels of education are welcome, from high school to university, and that all domains are welcome, including teacher training, engineer training, etc.

New rules for improving CAS capabilities when computing improper integrals. Applications in Math Education

José Luis Galán-García¹, Gabriel Aguilera-Venegas¹, Pedro Rodríguez-Cielos¹,
Yolanda Padilla-Domínguez¹, María Ángeles Galán-García¹

In many Engineering applications the computation of improper integrals is a need. In [1] we pointed out the lack of some CAS when computing some types of improper integrals. Even more, the work developed showed that some improper integrals can not be computed with CAS using their build-in procedures.

In this talk we will develop new rules to improve CAS capabilities in order to compute improper integrals such as:

$$1. \int_0^{\infty} f(x) g(x) dx \quad ; \quad \int_{-\infty}^0 f(x) g(x) dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) g(x) dx$$

where $g(x) = 1$ or $g(x) = \sin(ax)$ or $g(x) = \cos(ax)$ and $f(x) = \frac{p(x)}{q(x)}$ with degree of $p(x)$ smaller than degree of $q(x)$ and $q(x)$ with no real roots of order greater than 1.

$$2. \int_0^{\infty} x^{\alpha} f(x) dx \quad \text{where } \alpha \in \mathbb{R} \setminus \mathbb{Z} \quad \text{or} \quad -1 < \alpha < 0$$

We will show some examples of improper integrals that CAS as MATHEMATICA, MAPLE, DERIVE or MAXIMA can not compute. Using advance techniques as Laplace and Fourier transforms or Residue Theorem in Complex Analysis, we will be able to develop new rules schemes for these improper integrals.

We will also describe the conclusions obtained after using these new rules with our Engineering students when teaching Advanced Calculus.

Keywords: Improper Integrals, Rules Development, CAS, Engineering

References

- [1] José L. Galán-García, Gabriel Aguilera-Venegas, María Á. Galán-García, Pedro Rodríguez-Cielos, Iván Atencia-Mc.Killop. Improving CAS capabilities: New rules for computing improper integrals. Applied Mathematics and Computation. Volume 316, 1 January 2018, Pages 525-540.

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Teaching Partial Differential Equations with CAS

José Luis Galán-García¹, Pedro Rodríguez-Cielos¹, Yolanda Padilla-Domínguez¹, María Ángeles Galán-García¹, Gabriel Aguilera-Venegas¹, Ricardo Rodríguez-Cielos²

Partial Differential Equations (PDE) are one of the topics where Engineering students find more difficulties when facing Math subjects.

A basic course in Partial Differential Equations (PDE) in Engineering, usually deals at least, with the following PDE problems:

1. **Pfaff Differential Equations**, which consists on finding the general solution for:

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0$$

2. **Quasi-linear Partial Differential Equations**, which consists on finding the general solution for: $P(x, y, z) p + Q(x, y, z) q = R(x, y, z)$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

3. Using **Lagrange-Charpit Method** for finding a *complete integral* for a given general first order partial differential equation: $F(x, y, z, p, q) = 0$.

4. **Heat equation** which consists on solving the second order PDE:

$$\begin{cases} k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, & k > 0 & 0 < x < L & t > 0 \\ u(0, t) = 0 & & u(L, t) = 0 & t > 0 \\ u(x, 0) = f(x) & & 0 < x < L & \end{cases}$$

5. **Wave equation** which consists on solving the second order PDE:

$$\begin{cases} a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} & 0 < x < L & t > 0 \\ u(0, t) = 0 & u(L, t) = 0 & t \geq 0 \\ u(x, 0) = f(x) & \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) & 0 < x < L \end{cases}$$

6. **Laplace's equation** which consists on solving the second order PDE:

$$\left\{ \begin{array}{lll} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & 0 < x < a & 0 < y < b \\ \frac{\partial u}{\partial x} \Big|_{x=0} = 0 & \frac{\partial u}{\partial x} \Big|_{x=a} = 0 & 0 < y < b \\ u(x, 0) = 0 & u(x, b) = f(x) & 0 < x < a \end{array} \right.$$

In this talk we will describe how we introduce CAS in the teaching of PDE.

The tasks developed combine the power of a CAS with the flexibility of programming with it. Specifically, we use the CAS DERIVE. The use of programming allows us to use DERIVE as a Pedagogical CAS (PECAS) in the sense that we do not only provide the final result of an exercise but also display all the intermediate steps which lead to find the solution of a problem. This way, the library developed in DERIVE serves as a tutorial showing, step by step, the way to face PDE exercises.

In the process of solving PDE exercises, first-order Ordinary Differential Equations (ODE) are needed. The programs developed can be grouped within the following blocks:

- **First-order ODE:** separable equations and equations reducible to them, homogeneous equations and equations reducible to them, exact differential equations and equations reducible to them (integrating factor technique), linear equations, the Bernoulli equation, the Riccati equation, First-order differential equations and nth degree in y' , Generic programs to solve first order differential equations.
- **First-order PDE:** Pfaff Differential Equations, Quasi-linear PDE, Lagrange-Charpit Method for First-order PDE.
- **Second-order PDE:** Heat Equation, Wave Equation, Laplace's Equation.

In this talk we will introduce some improvements (redefinition of programs and more types of ODE and PDE) with respect to the talks given in previous ACA [1, 2] related with these topics. We will also remark the conclusions obtained after using these techniques with our Engineering students.

Keywords: ODE, PDE, DERIVE, CAS, PECAS, Engineering

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About the Bulgarian experience in organizing National Student Olympiad in Computer Mathematics

Penka Georgieva¹

In this paper the experience in organizing, conducting and participating in the National Student Olympiad in Computer Mathematics "Acad. Stefan Dodunekov" (CompMath) in Bulgaria is presented.

CompMath has been running since 2009. The first two olympiads were experimental. Since 2011 it is in the competitions calendar of Bulgarian universities. The idea is to promote the tools of computer algebra software and inspire the academic community to be part of this Olympiad [1].

CompMath is organized annually, it is held by a National Committee (NC) and hosted by a different university every year. A General Assembly of CompMath consists of the team leaders from the participating universities and meets at least once a year. Every university student enrolled in a bachelor or master degree programme at a Bulgarian or foreign university can participate individually. The participants are divided into groups: Group A - Mathematics, Informatics and Computer Science, Group B - Engineering and Natural Science. Those in group B are allowed to compete in Group A. In the last year there was one more group for school mathematics, again experimentally. Ranking is done within each Group. Up to 50 percents of the participants are awarded golden, silver or bronze medals by the NC in an approximate ratio of 1 : 2 : 3. The Organizing Committee issues certificates for the participants and team leaders [2].

Thirty problems from different areas of mathematics (Algebra, Analytic Geometry, Calculus, Differential Equations, Probability theory, etc.) have to be solved within 4 hours. Twenty of the problems can be solved using basic CAS functions, the remaining ten require advanced knowledge and skills.

The participants are free to choose the technology they prefer. They are allowed to use only one CAS, Matlab and also to combine two or more CAS, CAS and Matlab. Some students use only Mathematica, other - MatLab and MuPad, engineering students prefer Maple, Matlab and MuPaD.

Rapid changes in computer and information technology and the large number of mathematical software are prerequisites for gradual changes in the teaching and learning of university mathematics. The CompMath is a step towards these changes at Bulgarian universities.

Keywords: Computer mathematics, Olympiad, Bulgarian experience

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Do we take advantage of ICT when teaching maths at primary and secondary education levels? Do we teach maths as we should?

Eugenio Roanes-Lozano¹

My son is finishing primary education and I consider many of the school activities he has undertaken a waste of time. For example, does it make sense to divide by hand a twelve-digit number by a seven-digit number, or to factorize by hand an integer, two of whose factors are 71 and 107?

Although these are very specific cases, the exercises proposed in math classes at the primary and secondary education levels are not usually interesting, are sometimes even tedious and do not make the student love but hate maths. For example, in the case of factoring a number, the key is to understand what is being done and to know how to do it (algorithm), but it makes no sense to resolve uncomfortable cases when powerful computer algebra systems (CAS) are available in computers, calculators and even smartphones. In addition, the exercises proposed are many times disconnected from the real world (unlike many of those proposed in tests such as Pisa).

The use of ICT (at least in Spain) is many times restricted to “doing some research” on certain specific topics, but this is often a euphemism, since what the student many times does is just a “Google search”.

Almost twenty years ago a secondary schoolmate asked me which software to use, since the computers of his school had only the operating system and an office package and there were no funds for software. My answer was taxative: the CAS Maxima and the (then new) dynamic geometry system (DGS) GeoGebra. GeoGebra has spread considerably at secondary education level (overshadowing the pioneers Cabri Géomètre and The Geometer’s Sketchpad and the other DGS), but no CAS has clearly spread at this level. Possibly, the use of the latter has even decreased at this level for two reasons:

- Derive being discontinued
- the incorporation of algebraic capabilities by GeoGebra.

Despite the fact that since the 90s different theories have been developed about the use of mathematical software in education, such as the “Black Box / Whyte Box Principle” [1,2] and the “Mathematical Creativity Spiral” [3] (Buchberger), the “Scaffolding Principle” [4] (Kutzler) or the “Elevator Principle” [5] (Cabezas and Roanes), the teaching of primary and secondary education level mathematics continues to have a very low level of experimentality and the intensive use of mathematical software in the classes is exceptional.

Which can be the reasons for the limited use of ICT in the math classes and the persistence of tedious activities?

Keywords: Computational mathematics, Mathematical education

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Technology enhanced e-assessments in Calculus courses with application of CAS

Elena Varbanova¹

1. About the changing face of engineering education

As long as education can change, the world can change.

The technological development in the twenty-first century naturally and inevitably leads to the introduction of new tools into the university education. The experience in e-learning at the Technical University of Sofia (TUS) can be traced back to 1998. Numerous technology enhanced lectures, tutorials, laboratory classes and related textbooks have been created by lecturers in engineering, informatics and mathematics courses. Students and PhD students have been also involved through the development of theses on applications of Virtual Learning Environments (VLE) such as Moodle and ILIAS for design and implementation of effective didactic models. The outcomes of their research contribute to the changing face of engineering education.

Mathematics education has undergone a transformation based on applications of Computer Algebra Systems (CAS) [2], [3] and VLE. Though their simultaneous application is a challenge, it is a good opportunity for digitalizing mathematics education. Actually, there is "no way back", i.e. no education, progress and development without technology. Digital mathematics exists and digital resistance is not appropriate.

The focus of a dynamic unity of VLE and CAS in mathematics courses at TUS was on students' motivated, active, conscious and emotional participation in the teaching-learning-assessment (TLA) process. Assessment was an integral part of the TLA process and the three components were equally considered and mutually interrelated. But something was missing in the digital environment . . . It was e-assessments (diagnostic, continuous, formative, summative) provided with tools for authentication & authorship analysis in online and blended environments assuming the student would take the assessment at a distance. And, finally, a brave but necessary project TeSLA has arrived [5].

2. Re-design of Calculus courses within the framework of TeSLA project

Ne varietatem timeamus. /Do not be afraid of diversity./

Since January 2016 TUS is a partner university in the TeSLA project which conception /philosophy/ is built on a "general trust" that knows no time limits or national

boundaries and could fit to any system of higher education. The project aims to support and assure e-assessment processes in order to improve the trust level across students, teachers and institutions. According to TeSLA LOGO this system provides continuous and modular trust-based authentication & authorship analysis for e-assessments.

Concerning the educational aspects of TeSLA system it has to be mentioned that its implementation could give a significant added value as well: highly qualified and experienced teachers can design and develop a great diversity of e-assessment activities with purposeful and balanced application of the potential of the VLE and course related software [1], [4]. And these cannot be achieved in a face-to-face educational mode. Of course, teachers have to be aware of possible abuse of technology: they have to make what is important technology supported, rather than what is technology-supported important.

The undergraduate course Calculus 1 (Calculus of One Variable) was one of the piloted courses in TeSLA project. It is taught to students first year of study. The latter could be a reason they to prefer performing the activities in university computer labs, not at home, i.e. not at a distance. Within the framework of TeSLA project six tools for authentication & authorship analysis have been developed and tested in seven partner universities. Understandably, *Face Recognition (FR)* and *Keystroke Dynamics (KD)* were recommended to be tested as suitable for mathematics courses. Another two named *Voice Recognition* and *Plagiarism* are also useful for assessing students' capability to defend their individual and collaborative courseworks or to explain multi-step solutions. In the presentation a real (follow up) assessment activity monitored by TeSLa system will be demonstrated.

The students performed three activities: enrolment and two follow-up assessment activities. It was both challenging and exciting to re-design the TLA process, modify existing in-class tests and re-formulate questions. On one hand, in order to enhance the quality of assessment activities, we took advantage of the wide range of types of questions available in VLE Moodle and used the potential of a CAS-environment to create questions as well as to facilitate students in selection of approaches to find, interpret and check up answers/solutions. On the other hand, appropriate support materials were provided online and students could use also external resources.

Almost all the types of questions available in VLE Moodle were included in the assessment activities: True/False, Numerical, Multiple choice, Matching, Drag and drop into text, Drag and drop onto image, Select missing words, Essay /open answer questions/. A didactic system of questions/problems has been created. This diversity of questions and flexibility in answering them considerably contributed to enrichment and enhancement of assessment activities. They could serve as innovative assessment practices in mathematics education. During the assessment process students are allowed/required to use CAS and submit the produced CAS-protocol with solutions, explanations, interpretations and reflection on the results. In the presentation illustrative examples will be shown.

3. The synergies between VLE, CAS and TeSLA tools for enhancing university mathematics

Some things can't be explained, only experienced.

All the three components of the triad teaching-learning-assessment are to be considered in tandem and not focus on any one of them. Through a balanced integration of VLE and CAS in the design and development of assessment activities and related learning and support materials we aimed at helping students built up habits for Lower Order Learning and Higher Order Learning in accordance with the improved Bloom's taxonomy.

The follow up activities were performed by students for the purpose of formative e-assessment. They were used for both summative purposes in that 20% of the overall mark was allocated to these activities and formatively in that detailed feedback was provided. The latter appeared to be a motivation for students to better prepare themselves and avoid attempts for "helping" classmates or using illegal ways far achieving higher results. Universities have a mission to create a good sense of fairness and honesty for students as an important element of culture and to ensure action. And here TeSLA tools come to the rescue.

An e-assessment tool provides data to teachers about students. Having information per student, and also per "classroom", teachers could propose changes in courses and curriculum in order to help students acquire sustainable knowledge and develop required competencies. Independent individual/collaborative learning with self-assessment can also be monitored by TeSLA tools. On time provided feedback both online and offline, based on the individual learning trajectory, allows students make a desired progress in their own pace anytime and anywhere.

4. Conclusion

Challenge is energy of life.

Technology can "replace" hundreds of teachers but a powerful methodology and highly qualified teachers can give thousands of technologies vitality. Training of teachers should be the key concern of universities: in creation of e-assessment activities interrelated with the teaching and learning activities and according to the principle "What gets assessed is what gets taught". Like any other technological tool TeSLA system need a professional attitude to be taken to. Converting tools into effectively integrable instruments is a real question in education. In this sense there is still much to be done in order to assure trust-based e-assessments.

Educational technology (ET) is to be considered as

ET=Technology OF Education+Technology IN Education.

A proper utilization of technology in education requires a policy of support for research in the field of educational science, high quality software and teacher training. Together we need to go further to 4C = Challenging Changes in Curricula and Courses.

Keywords: mathematics education, e-assessment, CAS, VLE, TeSLA system

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Analyzing the "Calculator Effect" of Different Kinds of Software for School Arithmetics and Algebra

Rein Prank¹

The author of this paper realised the need to think about misuse effects when participating in two activities connecting Mathematics teaching and computers: addition of school-style step by step solutions to the output of CAS, and compiling a workbook that contains programming tasks extracted from School Mathematics.

The possibility of getting the answer and solution of arithmetic and algebra tasks from external sources is currently provided by four kinds of software:

1. **Spreadsheets.** We usually think that spreadsheets do decimal calculations. Solving examples from School Mathematics demonstrates an unexpected and undocumented mixture of decimal and algebraic calculations.
2. **Lightweight drill environments** for arithmetic and algebra. They have quite small calculator effect because usually they do not enable entering user-provided exercises.
3. **Step by step solution environments** (MathXpert, Aplusix, T-algebra). The key question is again whether the student gets right to enter initial expressions of the tasks.
4. **"Algebra calculators"**. There are a few dozen programs designed specifically for doing students' homework (producing solutions with necessary explanations) for almost all technical exercises of School Algebra. But very often the calculators implement textbook algorithms without any intelligence.

Item 4 means that availability of solutions in educational CAS (for example, in Geogebra) will not change the situation very much. In the first years, the CAS solutions will most likely have the same imperfections as the current algebra calculators.

Many school arithmetic and algebra tasks can be converted to programming tasks: long multiplication or division, reducing fractions, multiplication of polynomials. For routine tasks, programming does not replace exercising with something easier. The situation can be different when we come to the more original tasks in textbooks. A task of replacing stars with given numbers can lead to an interesting logical journey. However, stronger students in middle grades are perfectly capable of programming a brute-force solution search. It is important to think about ways of protecting the more interesting tasks in textbooks from such shortcuts.

Keywords: Calculator effect, Computer algebra, School Mathematics

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Student Attitudes toward Technology Use in Math Education

Karsten Schmidt¹

In the Faculty of Business and Economics at Schmalkalden University, the matrix algebra course of the bachelor program has been taught in the PC lab for many years (one or two students in front of a PC, instructor's PC connected to a projector). A Computer Algebra System (CAS) is used throughout the course. Students can install the CAS on their private PCs, and have access to it during the final exam in the PC lab (then, naturally, only one student per PC). Other courses, like Introduction to Mathematics, and Introduction to Statistics, are still taught in a traditional classroom setting (blackboard, overhead projector, and pocket calculators). At the beginning of the 2010/11 winter semester, a survey was carried out to investigate whether the students preferred traditional or technology-based courses in mathematics, and how well they coped with the technology. During the 2015/16 winter semester, a similar survey was carried out to check whether students' attitudes toward the use of technology in the teaching of mathematics have changed over time. In this presentation we will look at the key questions of the questionnaire, display descriptive statistics as well as charts of the variables generated from the gathered data, and analyze the effect that certain characteristics of the students (e.g. male vs. female, or students who like math vs. those who do not) have on their answers. The new results will be also compared to those found five years ago.

Keywords: Student attitude toward CAS, Survey of students, Changes over time

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Dynamic visualizations for network flow optimizations problems with Mathematica

Włodzimierz Wojas¹, Jan Krupa¹

Network flows as a problem domain is considered as a part of such mathematical disciplines as: graph theory, combinatorial optimization, mathematical programming or operation research. It is taught at universities in framework of different academic courses, for example: Graphs and networks, Optimization methods, Linear programming, Mathematical programming or Operation research. In the framework of network flows a number of optimizations problems are considered, such as: shortest path problem, maximum flow problem or minimum cost flow problem. Newer versions of Mathematica contain some functions dedicate to solve some network flows optimizations problems. In this talk, first we would like to present a few dynamic visualizations of network flows in pure, generalized and dynamic networks using Mathematica. Next, we will present visualizations for maximum flow problem and minimum cost flow problem

Keywords: network flows, network optimization, didactics of mathematics, mathematics education, CAS, Mathematica

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Using TI-Nspire for the financial education of future engineers

Hanan Smidi¹

For several years, and more precisely since September 2011, "L'École de technologie supérieure" (ÉTS) has chosen the TI-Nspire as the mandatory calculator for most, if not, all courses leading to an engineering degree.

One compulsory course in particular requires students to learn about the financial profitability of a project. The TI-Nspire offers all the basic financial functions from calculating the time value of money, depreciation of a loan, solving complex equations, graphic representations as well as programming functions for a quicker and easier access.

With its software, the lecturer can easily show examples while students can follow and do the problems directly on their handheld devices in the classroom.

There are many financial calculators available and mostly, Excel is widely used to present financial models. The TI-Nspire allows the user to solve most financial math situations on one device and hence become an unavoidable and powerful tool for the student.

This short presentation is an overview of the TI-Nspire's powerful use of financial math in the classroom.

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Accurate plotting in 3D: how to choose the mesh

David G. Zeitoun¹, Thierry N. Dana-Picard²

When studying continuity of 1-variable functions, a classical example is given by the function f such that $f(x) = \sin \frac{1}{x}$. this is an opportunity to show an example of Heine's Theorem:

Theorem: *Let f be a function defined on a punctured neighborhood of the real number a . The two following properties are equivalent:*

1. *The function f has a limit l at a (maybe infinite).*
2. *For every convergent sequence (x_n) of real numbers whose limit is a , the sequence $(f(x_n))$ has limit l .*

Take the sequences given by

$$x_n = \frac{1}{2n\pi} \text{ and } y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Both sequences converge and have a limit equal to 0, but $f(x_n)$ and $f(y_n)$ have different limits, showing that f has a discontinuity at 0.

This situation cannot be illustrated using a plot. Figure 1 shows the strange plot obtained using a standard command*. Zooming does not improve the visualization, as the function has an infinite number of oscillations within a compact interval around 0. Students have difficulties both with the visualization and with "for every sequence". The teacher has to insist on the fact that this theorem is mostly used to disprove continuity/existence of a limit.

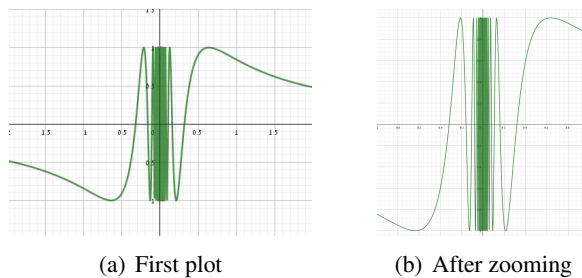


Figure 1: Strange plots for a one-variable function

*We used GeoGebra, but the same phenomenon appears with any other package.

Transition towards Calculus II, with functions of 2 real variables (or more than two) leads to problems, either different or more of the same. Sometimes, the visualization becomes harder to understand. For example, consider the function given by

$$f(x, y) = \frac{1}{1 - (x^2 + y^2)}. \quad (1)$$

Using standard commands for plotting graphs of 2-variable functions may provide a non accurate result (see Figure 2).

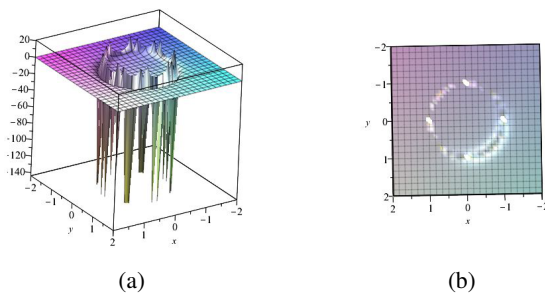


Figure 2: Strange plots for a two-variable function

The function has non-isolated singularities, but the plot in cartesian coordinates does not show them. Moreover the plot shows a lot of needles. The reason is that the standard `plot3d` command divided the given domain using a cartesian mesh, made of squares; this can be visualized when looking on the graph "from above", as shown in Figure 2b. Actually the plot is obtained using numerical computations: the CAS computes values of the function on the edges of the cells and make interpolations for what happens inside the cells, as explained in [2]. Zooming is useless, as this only inflates the cells, but does not compute new values for the function inside each cell.

Based on a prior mathematical analysis of the continuity of the function, a more accurate plot may be achieved using a new coordinate systems (See the reference [4]).

We choose new coordinates, which fit the specific coordinates of the given function. Figure 3a shows a plot of the function defined in Equation (1), using polar coordinates, with $x = r \cos \theta$, $y = r \sin \theta$. Even the choice of these coordinates do not ensure that the plot will be really accurate. Figure 3b and Figure 3c represent the same plot, viewed from different directions, after a slight modification of the domain. Here the interpolations once again hide the actual discontinuities.

Other modifications, such as considering $g(x, y) = 1/(3 - (x^2 + y^2))$, or $h(x, y) = 1/(1 - (x^2 + 3y^2))$ lead students to understand the need for a mathematical analysis of the data prior to computerized work.

Algorithms have been described in [3] in order for the software to determine a mesh which will provide an accurate plot. In our talk we will present both further advances in this field, and actual situations encountered in classroom. Moreover, we

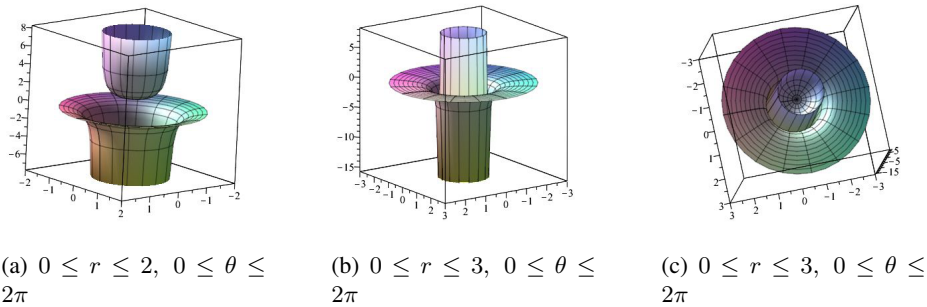


Figure 3: The influence of the choice of the plotting domain

can mention the possibility to enhance more understanding by using Virtual Reality, as described in [1].

Keywords: surface, accurate plot, mesh

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Addressing discrete mathematics problems in the classroom

A. Bergeron-Brlek¹

The TI-Nspire CX CAS calculator is mandatory in all mathematics courses at École de technologie supérieure. Every student has a handheld device in the classroom and can use it in real time. Making students work actively in the classroom is an effective way for improving their knowledge and understanding of the concepts.

The compulsory course Logic and Discrete Mathematics (MAT210) is given to software engineering students. Using a CAS in this course enables the teacher and the students to explore more complex examples. For instance, students can manipulate large prime numbers in the study of the RSA cryptographic system, or solve recurrence relations related to counting problems. We study, among other topics in this course, the complexity of algorithms.

In this talk, we will present our approach to handle this notion. Using Nspire, students are guided in the implementation of several algorithms to solve the same problem. In order to measure the time complexity (using the big- O notation), they run the algorithms on samples of different size and plot the results. This leads to a better understanding of the big- O notation, which is then confirmed algebraically using the handheld device.

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Analyzing discrete suspended chains using computer algebra

Gilbert Labelle¹

The mathematical description of the shape of various kinds of suspended chains, cables or funiculars under gravity is well covered in the scientific literature. In this talk we apply computer algebra to analyze, classify and animate suspended discrete chains whose links are “thin straight rods” joining the origin O to a variable endpoint P in the closed right half-plane. We use Lagrange multipliers to minimize the potential energy of each chain.

In contrast with the continuous limiting case of the catenary where the suspended chain is given explicitly (up to translation and zoom) as an arc of the hyperbolic cosine, the global shape of such discrete regular suspended chains has no simple explicit expression and falls into 3 classes :

Concave, Parallel, Convex,

according to the values of the Lagrange multipliers and the position of endpoint P . This provides to undergraduate students a stimulating example of the application of Lagrange multipliers and computer algebra methods to analyse a discrete optimization problem.

Keywords: Discrete chains, catenary, Lagrange multipliers, computer algebra

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Consolidation of abstract knowledge in the process of confronting errors using digital tools: The case of the inflection point

Anatoli Kouropatov¹, Regina Ovodenko²

At the previous ACA conference (see the reference [6]) we reported the results about the development of the integrated teaching unit that was designed towards learning an entire mathematical concept – the inflection point. The unit was built in the digital environment ([3]) and includes geogebra labs, interactive digital questionnaires, and videos, as well as a variety of investigative assignments that are based on them. This environment has been developed with special attention to addressing errors. The development of the environment was informed by research regarding the use of technological tools in math education and research about typical errors in specific mathematical subjects, such as functions ([2]), tangent lines ([1], [7], [9]), inflection ([8]), and so on. We theorized that learning with this unit would allow students to confront errors and to consolidate knowledge about the inflection point. With the purpose of testing this conjecture, we conducted a short feasibility study with a pair of first year students from the Industrial Engineering College. These students are considered advanced students (according to high formal achievements and their lecturer’s personal opinion). It was suggested to the students that they learn the unit after they learned the concept of the inflection point during the course Calculus 1. Their previous encounter with this concept consisted of the part of the process that dealt with investigating functions based on algorithmical usage of well-known theorems related to the concept. The study was organized as a two-hour clinical interview in laboratory conditions. The students’ work was documented and transcribed with the purpose of analyzing their learning process. The analysis of the students’ learning process has been conducted using “Abstraction in Context” (AiC) as developed in [5] as a theoretical framework and as methodological tool ([4]). According to “Abstraction in Context”, learners vertically reorganize previous elements of knowledge to construct new (for the learner) elements and to consolidate previously constructed (by the learner) elements. This construction/consolidation process takes place in students’ minds in a specific context, in our case – learning the inflection point concept using the digital-based teaching unit. In the conference we will present the methodology we used at the study as well as empirical evidence regarding the students’ learning process in general, and regarding the consolidation of abstract knowledge in the process of confronting errors using digital tools, in particular.

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Periodic and Nontrivial Periodic Input in Linear ODEs (Part I, Part II)

Michel Beaudin¹

Differential equations courses are among the ones where the use of computer algebra systems (CAS) was first experienced. In many cases, it was and it is still for solving application problems where the computation can become very long and tedious. Textbooks as [1] and [2] contain very interesting projects on which students can work. Unfortunately, some authors seem to forget the important role CAS can play in increasing student's understanding of theoretical concepts.

The first part of the talk will be devoted to introduce the subject: we will present the classical problem of finding the steady-state solution of a damped mass-spring problem where the external force is a pure cosine of different frequency. Then, in the second part, the external force will be a nontrivial periodic one. This is well documented but we rarely see different approaches. One approach we will use is the convolution of the input with the impulse response. Another approach will be the use of Fourier series because the linearity of the differential equation allows us to apply the principle of superposition. In both cases, the CAS will work for us, computing the convolution and finding the Fourier expansion.

Finally in the case of underdamping, the shape of the frequency response is best understood and illustrated with the aid of sliders. For this purpose, the TI-Nspire CX CAS software will be used. The notebook [3] contains many examples of how to use it in differential equations.

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Introducing parametric curves with CAS

Louis-Xavier Proulx¹

The TI-Nspire CX CAS handheld device is mandatory in every math courses at École de technologie supérieure. Our engineering students learn single-variable calculus in a one-term course (MAT145). Students get their motivation to learn mathematical concepts through applied problems. Hence, in-house course notes [1] were written to emphasize on the applications of the syllabus material and the use of CAS.

The multi-variable calculus course (MAT165) follows a classic reference textbook [2] contrary to the more hands-on approach pursued in MAT145. Many concepts can be explored and visualized with graphs of level surfaces. Plotting 3D objects is a useful feature of TI-Nspire, but it requires the use of parametric curves and surfaces. Vector functions are a fundamental tool for the course, but students hardly conciliate the graphical representations of these functions with their algebraic definition. Moreover, the students are challenged by the general and abstract setting in which these mathematical concepts are introduced.

The aim of this talk is to explore alternative ways for introducing the concept of parametric curves and vector functions using computer algebra. More focus will be given to different vector calculus concepts presented in applied problems. Software such as TI-Nspire will be used to graph and manipulate parametric curves.

Keywords: Parametric curves, Vector functions, Applications

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Visualizations of the nondominated set and the efficient set in multicriteria optimization problems using Mathematica

Włodzimierz Wojas¹, Jan Krupa¹

Multicriteria optimization also known as multicriteria programming is a sub-discipline of operation research. It is taught students in framework of such academic courses as for example: Operation research, Multiobjective optimization, Optimization methods or Mathematical programming. Multicriteria optimization problem has a general form:

$$f(x) = (f_1(x), f_2(x), \dots, f_k(x)) \rightarrow \min/\max$$

subject to $x \in X, X \subset \mathbb{R}^n$

where x is a decision variable vector, X is a feasible set in decision space \mathbb{R}^n , $(f_1(x), f_2(x), \dots, f_k(x))$ is a criterion vector and min or max are understood in accordance with the partial order P in criterion space \mathbb{R}^k . We define: a feasible set Y in criterion space as the image of the set X under $f = (f_1, f_2, \dots, f_k)$, the non-dominated set $Y_N = \{y \in Y : \text{there is not } y' \in Y \text{ with } y' P y\}$ and the efficient set $X_E = \{x \in X : f(x) \in Y_N\}$. Many academic books contain visualizations of sets X, Y, Y_N, X_E for some linear functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It would be more difficult but didactically useful to present these sets also for functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. It would rather require computer support using for example CAS programs. In this talk we would like to present a few didactic visualizations of sets X, Y, Y_N, X_E for some functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ using Mathematica.

Keywords: multicriteria optimization, multicriteria programming, didactics of mathematics, mathematics education, CAS, Mathematica

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Fractals and tessellations: from K's to cosmology

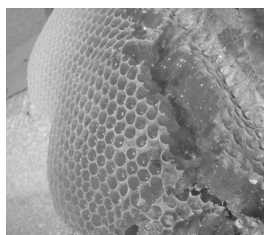
Thierry Dana-Picard¹, Sara Hershkovitz²

We present two related topics which can accompany Mathematics Education from early childhood to university and even beyond.

1. A *tessellation* is a partition of a space (usually a Euclidean space like the Euclidean plane or the Euclidean 3-dimensional space) by elements of a finite set, called tiles (more precisely, they are non-empty compacts). We begin with considering tilings by translations, i.e. two isometric tiles are deductible from one another by a translation (excluding rotations or symmetries). Generalization is possible to surfaces locally topologically equivalent to a plane. See Figure 1. Another generalization is to accept symmetries (either central or axial), as the



(a) Plane tessellation



(b) Honeycomb

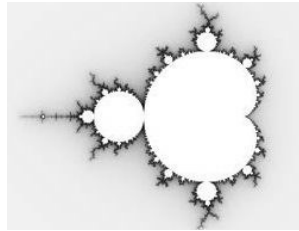
Figure 1: Tessellations

tessellation using a 4th generation Sierpinski triangle in Figure 3 (b).

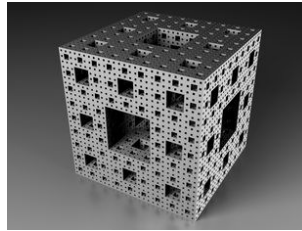
2. A *fractal* is an abstract object used to describe and simulate naturally occurring objects, showing self-similarity at increasingly small scales. Among the most known examples is the *Mandelbrot set* (Figure 1 (a)). An example of a 3D fractal is the so-called *Menger sponge*. See Figure 2

We show how to use a standard triangular grid to produce tessellation, and how to use a fractal to build a tessellation. Figure 3 shows a Sierpinski triangle, a plane tessellation built with it and a Sierpinski pyramid.

Different levels of technology can be used: low-tech such as paper and pencil, then progressive introduction of Dynamic Geometry (in our case GeoGebra: <http://geogebra.org>) and Computer Algebra Systems (we used Maple 2017), together with easy free software to work with images (Irfanview: <http://irfanview.com>).

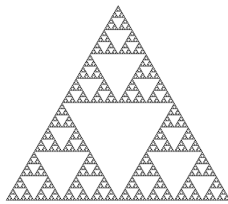


(a) Mandelbrot set

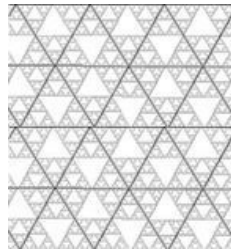


(b) Menger sponge

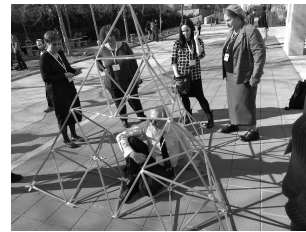
Figure 2: 2D and 3D fractals



(a) Sierpinski triangle



(b) Sierpinski tessellation



(c) Sierpinski pyramid

Figure 3: 2D and 3D fractals

We demonstrate how to work with technology, using GeoGebra applets, animations* and Maple programming. The examples serving as a basis can come from everyday life and also from more advanced scientific topics, such as the shape of space. In particular, we may quote the following works:

[(i)]Luminet's theory of *wrapped universe* [1] relies on a kind of 3D tessellation; see Figure 4 (a). A work presented at ACA 2017 in Jerusalem in the session on Applied Physics [2], describing the repartitions of galaxies using a Sierpinski gasket; see Figures 4 (b) and (c)[†].

Part of this exploration has been performed with in-service teachers in a special lab, last February.

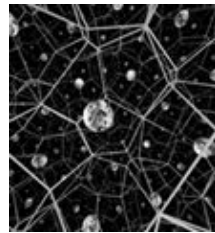
Keywords: Tessellations, Fractals, CAS, DGS

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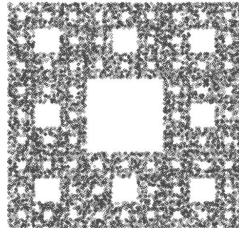
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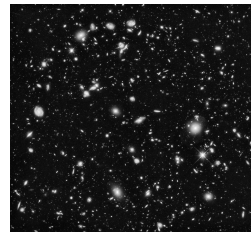
[†]Credit: NASA/Hubble



(a) Wraparound universe



(b) Sierpinski gasket



(c) Repartition of galaxies

Figure 4: The shape of space

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The Runge Example for Interpolation and Wilkinson’s Examples for Rootfinding

Leili Rafiee Sevyeri¹, Robert M. Corless²

We look at two classical examples in the theory of numerical analysis, namely the Runge example for interpolation and Wilkinson’s example (actually two examples) for rootfinding. We use the modern theory of backward error analysis and conditioning, as instigated and popularized by Wilkinson, but refined by Farouki and Rajan. By this means, we arrive at a satisfactory explanation of the puzzling phenomena encountered by students when they try to fit polynomials to numerical data, or when they try to use numerical rootfinding to find polynomial zeros. Computer algebra, with its controlled, arbitrary precision, plays an important didactic role.

Keywords: Interpolation, Rootfinding, Conditioning, Sensitivity.

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A non-iterative method for solving nonlinear equations

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Newton-Raphson method is the most commonly used iterative method for finding the root(s) of a real-valued function or nonlinear systems of equations. However, its convergence is often sensitive to the error in its initial estimation of the root(s). This talk will present a non-iterative method that mitigates non-convergence. An auxiliary initial-value problem of ordinary differential equation(s) is generated by a Computer Algebra System first, then integrated numerically over a closed interval. The solution(s) to the original systems of nonlinear equations is obtained non-iteratively at the end of the interval. A proof of the theorem serving as the base for this new method is presented at the talk. Several examples will illustrate its guaranteed convergence, a clear advantage over the Newton-Raphson method.

Keywords: Non-iterative method, Convergence, Nonlinear equations, Computer algebra

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What is the integral of x^n ?

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Thus, computer algebra systems usually solve a problem under the implied assumption that any parameters appearing in the problem will take values that lead to the general result returned. The title refers to the fact that the systems return the integral of x^n as $x^{n+1}/(n+1)$ while omitting the condition $n \neq -1$.

We shall use the word *specialization* to describe the action of substituting specific values (usually numerical, but not necessarily) into a formula. The *specialization problem* is a label for a cluster of problems associated with formulae and their specializations, the problems ranging from inelegant results to invalid ones. For example, in [2] an example is given in which the evaluation of an integral by specializing a general formula misses a particular case for which a more elegant expression is possible. The focus here, however, is on situations in which specialization leads to invalid or incorrect results. To illustrate the problems, consider

$$I_1 = \int \left(\alpha^{\sigma z} - \alpha^{\lambda z} \right)^2 dz = \frac{1}{2 \ln \alpha} \left(\frac{\alpha^{2\lambda z}}{\lambda} + \frac{\alpha^{2\sigma z}}{\sigma} - \frac{4\alpha^{(\lambda+\sigma)z}}{\lambda + \sigma} \right). \quad (1)$$

Expressions equivalent to this are returned by Maple, Mathematica and many other systems, such as the Matlab symbolic toolbox. It is easy to see that the specialization $\sigma = 0$ leaves the integrand in (1) well defined, but the expression for its integral on the right-hand side is no longer defined. If we pursue this further, we see that there are multiple specializations for which (1) fails, *viz.* $\alpha = 0$, $\alpha = 1$, $\lambda = 0$, $\sigma = 0$, $\lambda = -\sigma$, and combinations of these. The question of how or whether to inform computer users of these special cases has been discussed in the CAS literature many

times [1]. A list of every special case for (1) is as follows.

$$I_1 = \begin{cases} \frac{1}{2\lambda \ln \alpha} \left(\alpha^{2\lambda z} - \alpha^{-2\lambda z} - 4z\lambda \ln \alpha \right), & \begin{cases} \lambda + \sigma = 0, \\ \alpha \neq 0, \alpha \neq 1, \sigma \neq 0; \end{cases} \\ z + \frac{1}{2\lambda \ln \alpha} \left(\alpha^{\lambda z} (\alpha^{\lambda z} - 4) \right), & \begin{cases} \sigma = 0, \\ \alpha \neq 0, \alpha \neq 1, \lambda \neq 0; \end{cases} \\ z + \frac{1}{2\sigma \ln \alpha} \left(\alpha^{\sigma z} (\alpha^{\sigma z} - 4) \right), & \begin{cases} \lambda = 0, \\ \alpha \neq 0, \alpha \neq 1, \sigma \neq 0; \end{cases} \\ \text{ComplexInfinity}, & \begin{cases} \alpha = 0, \\ \Re(\lambda z) \Re(\sigma z) < 0; \end{cases} \\ \text{Indeterminate}, & \begin{cases} \alpha = 0, \\ \Re(\sigma z) \Re(\lambda z) \geq 0; \end{cases} \\ \frac{1}{2 \ln \alpha} \left(\frac{\alpha^{2\lambda z}}{\lambda} + \frac{\alpha^{2\sigma z}}{\sigma} - \frac{4\alpha^{(\lambda+\sigma)z}}{\lambda + \sigma} \right), & \text{otherwise, (generic case)}. \end{cases} \quad (2)$$

Expressions such as this will be called *comprehensive antiderivatives*. There are several questions surrounding such expressions. The first is whether comprehensive antiderivatives should be returned to users. A second question is how systems can compute such expressions. The automatic discovery of exceptional cases is not easy. A third question concerns *continuity with respect to parameters*.

We shall discuss why the expression

$$\int x^n dx = \frac{x^{n+1}}{n+1} - \frac{1}{n+1}$$

is better than the usual expression, and how we found it.

Keywords: Specialization problem, integration, parameters, continuity

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Familiarizing students with definition of Lebesgue measure using Mathematica - some examples of calculation directly from its definition

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“Young man, in mathematics you don’t understand things. You just get used to them” John von Neumann

In this talk we present some examples of calculation the Lebesgue measure of some subsets of \mathbb{R}^2 directly from definition. We cannot find such examples in the literature we know. We will consider the following subsets of \mathbb{R}^2 : $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x^2, 0 \leq x \leq 1\}$, $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \sin x, 0 \leq x \leq \pi/2\}$, $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \exp(x), 0 \leq x \leq 1\}$, $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \ln(1-2r \cos x+r^2), 0 \leq x \leq \pi\}$, $r > 1$. The aim of these examples is to familiarize students with the definition of Lebesgue measure. We calculate sums, limits and plot graphs and dynamic plots of needed sets and unions of rectangles sums of which volumes approximate Lebesgue measure of the sets, using Mathematica. The title of this talk is very similar to the title of author’s article [1] which deals with definition of Lebesgue integral but our talk deals with definition of Lebesgue measure instead. Using Mathematica or others CAS programs for calculation Lebesgue measure directly from its definitions, seems to be didactically useful for students because of the possibility of symbolic calculation of sums, limits - checking our hand calculations and plot dynamic graphs. Moreover we get students used not only to definition of Lebesgue measure but also to CAS applications generally.

The following definitions we will use in our talk (see [9, 3]):

Rectangles. A closed rectangle R in \mathbb{R}^d is given by the product of d one-dimensional closed and bounded intervals: $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $a_j \leq b_j$ are real numbers, $j = 1, 2, \dots, d$. In other words, we have $R = \{(x_1, \dots, x_d) \in \mathbb{R}^d : a_j \leq x_j \leq b_j \text{ for all } j = 1, 2, \dots, d\}$. We remark that in our definition, a rectangle is closed and has sides parallel to the coordinate axis. In \mathbb{R} , the rectangles are precisely the closed and bounded intervals, while in \mathbb{R}^2 they are the usual four-sided rectangles. In \mathbb{R}^3 they are the closed parallelepipeds.

We say that the lengths of the sides of the rectangle R are $b_1 - a_1, \dots, b_d - a_d$. The volume of the rectangle R is denoted by $\text{vol}(R)$, and is defined to be $\text{vol}(R) = (b_1 - a_1) \cdots (b_d - a_d)$.

Definition 1. (see [3, 7, 8, 9]) Let $(\mathbb{R}^2, \mathfrak{M}, m)$ be measure space, where \mathfrak{M} is σ -algebra of Lebesgue measurable subsets in \mathbb{R}^2 , and m - Lebesgue measure on \mathbb{R}^2 .

The measure m for any $A \in \mathfrak{M}$ is defined by the following formula:

$$m(A) = \inf \left\{ \sum_{j=1}^{\infty} \text{vol}(R_j) : A \subset \bigcup_{j=1}^{\infty} R_j, R_j \text{ is closed rectangle in } \mathbb{R}^2, j \in \mathbb{N} \right\}. \quad (1)$$

Keywords: Higher education, Lebesgue measure, Application of CAS, Mathematica, Mathematical didactics

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CAS in Teaching Basics of Stereoscopy

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We use Wolfram Mathematica in teaching the basics of stereoscopy. Each stereo image is represented either as a stereopair (for parallel and cross-eye views) or as an anaglyph image (to be viewed with red-cyan glasses). By changing the parallax of the left and right parts of the stereogram, one can move the image (or some part of it) in front of or behind the window frame. We demonstrate basic stereoscopic effects and typical mistakes made by beginners.

Soon after the invention of photography in 1839, a first suggestion for a stereo camera was given by Brewster in 1847. By 1860 viewing stereo photos was a popular pastime [1]. Later - in 20th century - interest waned. Recently, with the development of technology interest in 3D imaging has increased: 3D cameras are available; 3D lenses exist for cameras and smartphones; 3D cartoons and movies can be watched on an ordinary tv or computer screen using anaglyph (red-cyan) glasses; there are special 3D TVs, 3D movie cinemas, glasses-free 3D displays, etc.; and, finally, the recent development of virtual reality.

Here, we demonstrate some basic tricks one can do with changing the position of the stereo window and basic mistakes of beginners (and not only beginners - similar mistakes can be seen in big-screen, multi-million-dollar budget movies!). For example, one of the standard "Wow!" effects with 3D images is when most of the image appears behind the frame, but part of the image is hanging out in front. One has to be careful: that part shouldn't be cut by the frame! It is especially important in action scenes when objects which are moving around can easily be cut by the frame border.

Wolfram Mathematica provides good tools for image processing as well as powerful analytic tools, 3D graphics and animation. These allow one to easily construct simple stereograms; demonstrate effects and distortions introduced by the incorrect methods of constructing stereo image; and manipulate constructed images to demonstrate different effects. We use Mathematica to demonstrate correct and incorrect "Wow!" effect realizations of the type described above. By changing relative parallax of left and right images, we change position of the image with respect to the frame. Our manipulations with stereo images we base on a classic textbook by J.G. Ferwerda [1].

Keywords: Stereoscopy, 3D Imaging

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