SPECIAL SESIONS

Applications of Computer Algebra - ACA2018

June 18–22, 2018

Santiago de Compostela, Spain

S6
Computational Differential and Difference Algebra

Thursday

Thu 21st, 11:30 - 12:00, Aula 9 – Eli Amzallag:
Bounds for Proto-Galois Groups

Thu 21st, 12:00 - 12:30, Aula 9 – Vladimir V. Bavula :
The global dimension of the algebras of integro-differential operators and their factor algebras

Thu 21st, 12:30 - 13:00, Aula 9 – Paweł Bogdan:
Effective calculation in studying the Jacobian Conjecture

Thu 21st, 16:30 - 17:00, Aula 9 – Alexander Levin:
Dimension Polynomials and the Einstein's Strength of Some Systems of Quasi-linear Algebraic Difference Equations

Thu 21st, 17:30 - 18:00, Aula 9 – Sebastian Falkensteiner:
Formal Power Series Solutions of First Order Autonomous Algebraic Ordinary Differential Equations

Thu 21st, 18:00 - 18:30, Aula 9 – Wei Li:
Computation of differential Chow forms for ordinary prime differential ideals

Thu 21st, 18:30 - 19:00, Aula 9 – Dmitry Lyakhov:
Group Classification of ODEs: a Challenge to Differential Algebra?

Thu 21st, 19:00 - 19:30, Aula 9 – Daniel Robertz:
Power series solutions of systems of nonlinear PDE's

Organizers

Vladimir Gerdt:
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Aim and cope

Algebraic differential and difference equations and systems of such equations arise in many areas of mathematics, natural sciences and engineering. One can say that difference equations relate to differential equations as discrete mathematics relates to continuous mathematics. Differential / difference computer algebra studies algebraic differential / difference equations in a constructive way that extends the methods and algorithms of commutative algebra and algebraic geometry. The main goal of the session is to consider the computational problems in differential/difference algebra to explore new constructive ideas and approaches oriented toward various applications.

Expected topics of presentations include (but are not limited to):

- Differential and Difference Equations and Systems
- Differential and Difference Gröbner (Standard) and Involutive Bases
- Differential and Difference Characteristic Sets
- Triangular Decompositions of Differential and Difference Systems
- Differential and Difference Elimination
- Algorithmic Generation of Finite Difference Approximations to PDEs
- Consistency and Stability Analysis of Finite Difference Approximations
- Difference-Differential Polynomials and Systems
- Software Packages for Differential and Difference Algebra
- Applications of Differential and Difference Algebra in Mathematics and Natural Sciences

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Bounds for Proto-Galois Groups

Eli Amzallag[1], **Andrei Minchenko**[2], **Gleb Pogudin**[3]

In studying linear differential equations of the type $Y' = AY, A \in M_n(C(t))$, it is often important to investigate the algebraic or differential relations among the solutions. The benefit of obtaining such data is that it can be used to anticipate the computational power needed to express solutions. In [4], Kolchin made this precise by establishing a link between how solutions to $Y' = AY$ might be expressed and different properties of the corresponding differential Galois group, an object he constructed exactly to capture relations among the solutions. These differential Galois groups can be realized as linear algebraic groups. In fact, many algorithms to compute them have been developed since Kolchin's foundational discussions of these results in [4] and [5].

Kovacic [6] proposed an algorithm for second-order differential equations. Compoint and Singer also provided an algorithm in [1] that can be applied to equations of any order, if it is known in advance that the differential Galois group is reductive. A general algorithm for computing the differential Galois group was designed by Hrushovski [3]. Making this algorithm practical and understanding its complexity is an important challenge. Hrushovski conjectured that none of its steps would "require more than doubly exponential time." In [2], Feng expounded on the details of Hrushovski's original algorithm with differential-algebraic terminology and improved the algorithm. He also formally defined an object computed in the first step of the algorithm, a proto-Galois group. Such a group is an algebraic group, contains the differential Galois group, and the computation of it allows one to reduce the computation of the differential Galois group to the hyperexponential case, which is addressed by the algorithm in [1]. In Hrushovski's algorithm, a proto-Galois group is computed by making an ansatz based on an a priori bound for the degrees of defining polynomials of the group. Thus, such a bound is an essential part of the algorithm. Moreover, it also needed for understanding the complexity of the algorithm.

In [2], Feng showed that there exists a proto-Galois group defined by polynomials of degree at most sextuply exponential in n. Sun [7] utilized triangular sets in place of the Groebner bases used by Feng. This different choice of representation for a group leads to a bound triply exponential in n.

We adopt a different emphasis from both Feng and Sun. Instead of focusing on equations for the group's corresponding radical ideal, we take a more geometric approach and focus on equations that define a proto-Galois group as an algebraic variety in $GL_n(C)$. In conjunction with exploiting the structural theory of algebraic groups, this approach allows us to further improve on Feng's bound and thereby improve the algorithm. We obtain an explicit bound of the form $n^{O(n^4)}$.

We also assess the practicality of using Hrushovski's algorithm for $n = 2, 3$, the cases that arise most often in applications. We expect to determine tighter bounds than our general bound suggests for these cases. In fact, we have established a tighter bound for $n = 2$, for which our final bound is 6. We will discuss how we obtained this result. We will also discuss work in progress for $n = 3$ and extending our methods for $n = 2$ to those cases.

Keywords: Algebraic Geometry, Group Theory and Generalizations, Ordinary Differential Equations

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The global dimension of the algebras of integro-differential operators and their factor algebras

V. V. Bavula[1]

We discuss some homological properties of the algebras of integro-differential operators and their factor algebras. In particular, their global dimension and weak homological dimensions are found.

Keywords: the algebra of integro-differential operators, the weak homological dimension, the global dimension

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Effective calculation in studying the Jacobian Conjecture

Paweł Bogdan[1]

In 1930s Keller stated a problem which is known nowadays as the Jacobian Conjecture. In [1] Crespo and Hajto gave an equivalent condition to this Conjecture in a language of Picard-Vessiot theory. They also gave an effective criterion to determine whether a given polynomial map is an automorphism. Their result was improved in [2].

The work on this improvement led me to propose a method to invert polynomial maps $F = (F_1, ..., F_n)$ on a field K such that, for every $i \in \{1, ..., n\}$ $F_i = X_i + H_i$, where H_i has a vanishing order at least 2. My algorithm does not perform derivatives neither division so it can be applied to maps over finite fields. A description of the algorithm can be found in [3] and an estimation of its computational complexity can be found in [4].

In my talk I will present the algorithm and the estimation of its complexity. I will also discuss effective implementations of it on various Computer Algebra Systems.

Keywords: polynomial automorphisms, Jacobian Conjecture, algorithmics

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Formal Power Series Solutions of First Order Autonomous Algebraic Ordinary Differential Equations*

Sebastian Falkensteiner[1], **J.Rafael Sendra**[2]

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Given a first order autonomous algebraic ordinary differential equation, i.e. an equation of the form

$$F(y, y') = 0 \text{ with } F \in \mathbb{K}[y, y'],$$

we present a method to compute all formal power series solutions. Furthermore, by choosing for instance $\mathbb{K} = \mathbb{C}$, the computed formal power series solutions are indeed convergent in suitable neighborhoods.

We follow the algebro-geometric approach by Feng and Gao [2] and consider y and y' as independent variables, let us say y and z. Then F implicitly defines an affine plane curve where local parametrizations can be computed, see e.g. [3]).

We show a sufficient and necessary condition on such a local parametrization to obtain a formal power series solution of the original differential equation by substitution. Moreover, we present a polynomial-time algorithm for computing all the initial tuples, i.e. the first two coefficients of a formal power series, which can be extended to a solution. By choosing a particular initial tuple, a second algorithm determines the coefficients of all solutions starting with this initial tuple up to an arbitrary order.

A full version [1] has been submitted to a journal and is online available.

Keywords: Algebraic autonomous differential equation, algebraic curve, local parametrization, formal power series solution

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Dimension Polynomials and the Einstein's Strength of Some Systems of Quasi-linear Algebraic Difference Equations

Alexander Evgrafov[1], **Alexander Levin**[2]

We present a difference algebraic technique for the evaluation of the Einstein's strength of quasi-linear partial difference equations and some systems of such equations. Our approach is based on the properties of difference dimension polynomials that express the Einstein's strength and on the characteristic set method for computing such polynomials. The obtained results are applied to the comparative analysis of difference schemes for some chemical reaction-diffusion equations.

Keywords: Difference dimension polynomial, Autoreduced set, Einstein's strength

1 Preliminaries

Let K be an inversive difference field with a basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_m\}$ and Γ the free commutative group generated by σ. If $\gamma = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma$, then the number $\operatorname{ord}\gamma = \sum_{i=1}^{m} |k_i|$ is called the *order* of γ; if $r \in \mathbb{N}$, we set $\Gamma(r) = \{\gamma \in \Gamma \mid \operatorname{ord}\gamma \leq r\}$. In what follows we denote the set $\{\alpha_1, \dots, \alpha_m, \alpha_1^{-1}, \dots, \alpha_m^{-1}\}$ by σ^* and use the prefix σ^*- instead of "inversive difference". A reflexive difference ideal will be refer to as a σ^*-ideal.

Let $R = K\{y_1, \dots, y_n\}^*$ be the ring of σ^*-polynomials in n σ^*-indeterminates over K. (As a ring, $R = K[\{\gamma y_i \mid \gamma \in \Gamma,\, 1 \leq i \leq n\}]$) An n-tuple $\xi = (\xi_1, \dots, \xi_n)$ with coordinates in some σ^*-overfield K' of K is said to be a solution of the set of σ^*-polynomials $F = \{f_j \mid j \in J\} \subseteq R$ or a solution of the system of algebraic difference equations

$$f_j(y_1, \dots, y_n) = 0 \;\; (j \in J) \tag{1}$$

if F is contained in the kernel of the natural difference K-homomorphism ("substitution") $R \to K'$ ($y_i \mapsto \xi_i$). The system (1) is called *prime* if the σ^*-ideal P generated by the set F in R (it is denoted by $[F]^*$) is prime. In this case the quotient field L of R/P has a natural structure of a finitely generated σ^*-field extension of K: $L = K\langle \eta_1, \dots, \eta_n\rangle^*$ where η_i is the canonical image of y_i in L. (As a field, $L = K(\{\gamma(\eta_i) \mid \gamma \in \Gamma,\, 1 \leq i \leq n\})$.) As it is proven in [3, Section 6.4], there exists a polynomial $\phi_{\eta|K}(t) \in \mathbb{Q}[t]$ such that

$\phi_{\eta|K}(r) = \operatorname{tr.deg}_K K(\{\gamma\eta_j | \gamma \in \Gamma(r), 1 \leq j \leq n\})$ for all sufficiently large $r \in \mathbb{Z}$.

This polynomial is called the σ^*-*dimension polynomial* of the σ^*-field extension L/K associated with the system of σ^*-generators $\eta = \{\eta_1, \dots, \eta_n\}$. It is also said to be the σ^*-dimension polynomial of system (1). We refer to [3, Chapter 6] and [4, Chapters 4 and 7] for properties, invariants, and methods of computation of σ^*-dimension polynomials.

Let us consider a system of equations in finite differences with respect to unknown functions of m real (or complex) variables $x_1, \dots, x_m$ that induces a prime system of algebraic difference equations. (The m basic automorphisms are defined by the shifts of the arguments: for any function $g(x_1, \dots, x_m)$, $\alpha_i : g(x_1, \dots, x_m) \mapsto g(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_m)$ where $h_1, \dots, h_m$ are some real (or complex) numbers.) It is shown in [4, Section 7.7] that the σ^*-dimension polynomial of such a system expresses its strength in the sense of A. Einstein. This important characteristic of the system is a difference counterpart the concept of strength of a system of PDEs introduced in [1], see [4, Section 7.7] for details.

2 Autoreduced sets of quasi-linear σ^*-polynomials. Computation of the Einstein's Strength

With the above notation, let $\Gamma Y = \{\gamma y_i | \gamma \in \Gamma, 1 \leq i \leq n\} \subseteq R$; the elements of this set are called *terms*. The order $\operatorname{ord} u$ of a term $u = \gamma y_j$ is defined as the order of γ.

In what follows we consider the set $\mathbb{Z}^m$ as the union of 2^m *orthants* $\mathbb{Z}_j^{(m)}$ $(1 \leq j \leq 2^m)$, that is, Cartesian products of m factors each of which is either $\mathbb{N} = \{k \in \mathbb{Z}, k \geq 0\}$ or $\bar{\mathbb{Z}}_- = \{k \in \mathbb{Z}, k \leq 0\}$. We set $\Gamma_j = \{\alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma \,|\, (k_1, \dots, k_m) \in \mathbb{Z}_j^{(m)}\}$ and $(\Gamma Y)_j = \{\gamma y_i \,|\, \gamma \in \Gamma_j, 1 \leq i \leq n\}$, so that $\Gamma Y = \bigcup_{j=1}^{2^m} (\Gamma Y)_j$. A term $v \in \Gamma Y$ is called a *transform* of a term $u \in \Gamma Y$ if u and v belong to the same set $(\Gamma Y)_j$ and $v = \gamma u$ for some $\gamma \in \Gamma_j$. We also fix an *orderly ranking* on ΓY, that is, a well-ordering $\leq$ of ΓY such that
(i) If $u \in (\Gamma Y)_j$ and $\gamma \in \Gamma_j$, then $u \leq \gamma u$; (ii) If $u, v \in (\Gamma Y)_j$, $u \leq v$ and $\gamma \in \Gamma_j$, then $\gamma u \leq \gamma v$; (iii) If $u, v \in \Gamma Y$ and $\operatorname{ord} u < \operatorname{ord} v$, then $u < v$.

If $A \in R$, then the greatest (with respect to $\leq$) term in A is called the *leader* of A; it is denoted by u_A. If $d = \deg_{u_A} A$ and A is written as a polynomial in u_A, then the coefficient of u_A^d is called the *initial* of A and is denoted by I_A. If $d = 1$ then the σ^*-polynomial A is called *quasi-linear*.

Let $A, B \in R$. The σ^*-polynomial A is said to be *reduced* with respect to B if A does not contain any power of a transform γu_B whose exponent is greater than or equal to $\deg_{u_B} B$. If $\mathcal{A} \subseteq R \setminus K$, then a σ^*-polynomial $A \in R$, is said to be reduced with respect to $\mathcal{A}$ if A is reduced with respect to every element of $\mathcal{A}$. A set $\mathcal{A} \subseteq R$ is said to be *autoreduced* if either $\mathcal{A} = \emptyset$ or $\mathcal{A} \bigcap K = \emptyset$ and the elements of $\mathcal{A}$ are reduced with respect to each other. As it is shown in [3, Section 3.4], distinct elements of an autoreduced set $\mathcal{A}$ have distinct leaders and every autoreduced set is finite. Furthermore, if $A \in R$, then there exists a σ^*-polynomial $B \in R$ such that B is reduced with respect to $\mathcal{A}$ and $IB \equiv A(mod[\mathcal{A}]^*)$ where I is a product of transforms of initials of elements of $\mathcal{A}$. (We say that A *reduces to* B *modulo* $\mathcal{A}$.)

Let $A, B \in R$. We say that A has higher rank than B and write $\operatorname{rk} A > \operatorname{rk} B$ if either $A \notin K$, $B \in K$, or $u_B < u_A$, or $u_A = u_B$ and $\deg_{u_A} B < \deg_{u_A} A$. If $u_A = u_B$ and $\deg_{u_A} A = \deg_{u_A} B$, we say that A and B have the same rank and write $\operatorname{rk} A = \operatorname{rk} B$. Assuming that elements of an autoreduced set in R are arranged in the order of increasing rank, we compare such sets as follows: if $\mathcal{A} = \{A_1, \dots, A_p\}$ and $\mathcal{B} = \{B_1, \dots, B_q\}$, then $\mathcal{A}$ is said to have lower rank than $\mathcal{B}$ if either there exists $k \in \mathbb{N}$, $1 \leq k \leq \min\{p, q\}$, such that $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i < k$ and $\operatorname{rk} A_k < \operatorname{rk} B_k$, or $p > q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, q$.

By [3, Proposition 3.4.30], every nonempty family of autoreduced sets contains an autoreduced set of lowest rank. If P is an ideal of R, then an autoreduced subset of P of lowest rank is called a *characteristic set* of P. Basic properties of characteristic sets are described in [4, Section 2.4]. In particular, it is shown that if P is generated by the σ^*-polynomials in the left-hand sides of a prime system of difference equations (1) and $\mathcal{A}$ is a characteristic set of P, then the σ^*-dimension polynomial of the system is determined by the leaders of elements of $\mathcal{A}$. Therefore, the strength of a prime system of difference equations is determined by a characteristic set of the associated σ^*-ideal in the ring of σ^*-polynomials.

An autoreduced subset $\mathcal{A}$ of R consisting of quasi-linear σ^*-polynomials is called *coherent* if it satisfies the following two conditions: (i) If $A \in \mathcal{A}$ and $\gamma \in \Gamma$, then γA reduces to zero modulo $\mathcal{A}$; (ii) If $A, B \in \mathcal{A}$ and $v = \gamma_1 u_A = \gamma_2 u_B$ is a common transform of u_A and u_B, then the σ^*-polynomial $(\gamma_2 I_B)(\gamma_1 A) - (\gamma_1 I_A)(\gamma_2 B)$ reduces to zero modulo $\mathcal{A}$.

The following two statements are the main results that allow one to evaluate the Einstein's strength of difference equations that arise from difference schemes for some chemical reaction-diffusion equations arising in many problems of transfusion, see [2].

Theorem 1. *If a characteristic set $\mathcal{A}$ of some σ^*-ideal in R consists of quasi-linear σ^*-polynomials, then $\mathcal{A}$ is a coherent autoreduced set. Conversely, if $\mathcal{A}$ is a coherent autoreduced set consisting of quasi-linear σ^*-polynomials, then it is a characteristic set of* $[\mathcal{A}]^*$.

Theorem 2. *Let $\preccurlyeq$ be a preorder on R such that $A_1 \preccurlyeq A_2$ iff u_{A_2} is a transform of u_{A_1}. Let A be a quasi-linear σ^*-polynomial and $\Gamma A = \{\gamma A \,|\, \gamma \in \Gamma\}$. Then the σ^*-ideal $[\mathcal{A}]^*$ is prime and all minimal (with respect to $\preccurlyeq$) elements of ΓA form a characteristic set of* $[\mathcal{A}]^*$.

Using the last two theorems and the expression of the σ^*-dimension polynomial given in [3, Theorem 6.4.8], we obtain σ^*-dimension polynomials that express the Einstein's strength of difference schemes for some quasi-linear reaction-diffusion PDEs (e. g., the Murray's equation and its particular cases), the system of PDEs of chemical reaction kinetics with the diffusion phenomena and the mass balance PDEs of chromatography. The results of the corresponding computations allow one to do comparative analysis of alternative difference schemes from the point of view of their strength.

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Computation of differential Chow forms for ordinary prime differential ideals

Wei Li[1], Ying-Hong Li[1]

The differential Chow form is an important associated form for a prime differential ideal or an order-unmixed differential cycle [1]. For example, it can characterize invariants of its corresponding prime differential ideal, such as the differential dimension, order, leading differential degree and differential degree. So it is desirable to devise efficient algorithms to compute the differential Chow form. In this talk, we propose algorithms for computing differential Chow forms for ordinary prime differential ideals which are given by characteristic sets. The algorithms are based on an optimal bound for the order of a prime differential ideal in terms of a characteristic set under an arbitrary ranking, which shows the Jacobi bound conjecture holds in this case. That is, $\mathrm{ord}(\mathrm{sat}(\mathcal{A})) \leq \mathrm{Jac}(\mathcal{A})$. Apart from the order bound, we also give a Bézout type degree bound for the differential Chow form. The computational complexity of the algorithms is single exponential in terms of the Jacobi number, the maximal degree of the differential polynomials in a characteristic set, and the number of variables.

Keywords: Differential Chow form, Jacobi bound, Single exponential algorithm

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Group Classification of ODEs: a Challenge to Differential Algebra?

Dmitry Lyakhov[1], **Vladimir Gerdt**[2], **Dominik Michels**[3]

One of the most prominent application of differential algebra is algebraic analysis of determining system of partial differential equations for infinitesimal symmetry generators. It provides receipts and software tools to compute the integrability conditions, to simplify (e.g. to interreduce) the system, to determine a dimension of its space, to construct the abstract Lie algebra for the symmetry generators, to apply the Lie symmetry algebra for ordinary differential equations (ODEs) to detect their linearizability [1] by point transformations. The problem of group classification for differential equations was first posed by the Norwegian mathematician Sophus Lie, the inventor of the concept and theory of continuous groups and their application to differential equations [2]. Lie began to solve the group classification problem for the second-order ordinary equation $y'' = f(x, y, y')$ and proved that this class of equations admits no more than an eight-parameter transformation group on the plane with the maximum size of the group is reached iff the equation is linear or equivalent to the linear one. The Russian mathematician Lev Ovsyannikov [3] proposed the equivalence transformation (ET) method for group classification and later [4] applied it to the ODE of form $y'' = f(x, y)$. The ET method is based on the fact that equivalent equations admit similar groups and ET is a similarity transformation. The problem of group classification admits reformulation as an elimination problem in differential algebra. However, even reproduction of the results, obtained in [4] by hand computation, seems to be too hard for the modern differential elimination tools. In the talk we discuss both the pure mathematical and computational issues of the group classification for ODEs.

Keywords: Differential Algebra, Group Classification, Ordinary Differential Equations

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Power series solutions of systems of nonlinear PDEs

Daniel Robertz[1]

One of the first existence theorems for a large class of PDEs is the Cauchy-Kovalevskaya Theorem [5]. In work of C. Méray and C. Riquier in the second half of the 19th century a generalization of the Cauchy-Kovalevskaya Theorem was obtained. Riquier's Existence Theorem asserts the existence of analytic solutions for the class of orthonomic and passive systems of PDEs [7, Chap. VIII]. J. M. Thomas [9] showed that polynomially nonlinear systems of PDEs can be decomposed into finitely many so-called simple differential systems, each of which can be solved for the highest ranked derivatives to obtain orthonomic and passive systems. Building also on work by M. Janet [4], the algorithmic details of the Thomas decomposition method have been recently developed [1], [2], [6], [8].

In this talk we explain how the differential Thomas decomposition can be used to find all power series solutions around sufficiently generic points of a system of nonlinear partial differential equations. Further applications of the Maple package for computing Thomas decompositions [3], e.g. to differential elimination, are demonstrated as well. The talk is based on joint work with Vladimir Gerdt and Markus Lange-Hegermann.

Keywords: completion to involution, Thomas decomposition, differential elimination

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