SPECIAL SESIONS

Applications of Computer Algebra - ACA2018

June 18–22, 2018

Santiago de Compostela, Spain

S7

Algebraic and Algorithmic Aspects of Differential and Integral Operators Sesión 02

Tuesday

Tue 19th, 15:30 - 16:00, Aula 9 – Vladimir Bavula:
The Jacobian algebras, their ideals and automorphisms

Tue 19th, 16:00 - 16:30, Aula 9 – François Boulier:
On the Parameter Estimation Problem for Integro-Differential Models

Tue 19th, 16:30 - 17:00, Aula 9 – Francisco-Jesús Castro-Jiménez:
Parametric b-functions for some hypergeometric ideals

Tue 19th, 17:30 - 18:00, Aula 9 – Cyrille Chenavier:
Reduction operators and completion of linear rewriting systems

Tue 19th, 18:00 - 18:30, Aula 9 – Yi Zhang:
Desingularization in the q-Weyl algebra

Tue 19th, 18:30 - 19:00, Aula 9 – David G. Zeitoun:
Solution of non-homogenous Ordinary Differential Equations using Parametric Integral Method

Wednesday

Wed 20th, 10:00 - 10:30, Aula 9 – Maximilian Jaroschek:
Low-Order Recombinations of C-finite Sequences

Wed 20th, 10:30 - 11:00, Aula 9 – Alexander Levin:
Some Properties and Invariants of Multivariate Difference-Differential Dimension Polynomials

Wed 20th, 11:30 - 12:00, Aula 9 – Jean-François Pommaret:
Computer algebra and the Lanczos problems in arbitrary dimensions

Wed 20th, 12:00 - 12:30, Aula 9 – Jamal Hossein Poor :
Symbolic computation for integro-differential-time-delay operators with matrix coefficients

Wed 20th, 12:30 - 13:00, Aula 9 – Clemens Raab:
Algebraic proofs of operator identities

Wed 20th, 13:00 - 13:30, Aula 9 – Bruno Salvy:
Definite Integration of D-finite Functions via Generalized Hermite Reduction

Thursday

Thu 21st, 10:00 - 10:30, Aula 9 – Thomas Dreyfus:
Effective criterion to test differential transcendence of special functions

Thu 21st, 10:30 - 11:00, Aula 9 – Sette Diop:
Observability and orders of derivatives of data

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Aim and cope

The algebraic/symbolic treatment of differential equations is a flourishing field, branching out in a variety of subfields committed to different approaches. In this session, we want to give special emphasis to the operator perspective of both the underlying differential operators and various associated integral operators (e.g. as Green's operators for initial/boundary value problems).

In particular, we invite contributions in line with the following topics:

- Symbolic Computation for Operator Algebras
- Factorization of Differential/Integral Operators
- Linear Boundary Problems and Green's Operators
- Initial Value Problems for Differential Equations
- Symbolic Integration and Differential Galois Theory
- Symbolic Operator Calculi
- Algorithmic D-Module Theory
- Rota-Baxter Algebra
- Differential Algebra
- Discrete Analogs of the above

- Software Aspects of the above

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Santiago de Compostela, June 18–22, 2018

The Jacobian algebras, their ideals and automorphisms

V. V. Bavula[1]

The talk is about general properties of the Jacobian algebras (in arbitrary many variables), classifications of their ideals, an explicit description of their groups of automorphisms. Explicit values of their global and weak dimensions are found.

Keywords: Jacobian algebra, group of authomorphisms, global and weak dimension

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Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

On the Parameter Estimation Problem for Integro-Differential Models*

François Boulier[1]

This talk summarizes a joint work with modelers and biologists [2]. It deals with the parameter estimation problem for dynamical systems presented by explicit systems of polynomial integro-differential equations (IDE).

Models formulated by means of IDE are very interesting because they are much more expressive than their ODE counterparts: they naturally permit to express delays (IDE are viewed as continuous delay differential equations in [9]), to take into account the age of populations (typical motivation for integral equations in population dynamics), to incorporate curves obtained by interpolating experimental data as integral kernels (an important feature for modeling processes interacting with complicated environment) and to handle non smooth (e.g. piecewise defined) inputs. See [6] and references therein.

The rest of this abstract is essentially borrowed from the introduction of [2].

IDE modeling raises, in turn, the problem of estimating parameters from experimental data. This talk focuses on a particular method, called the "input-output (IO) ideal" method, which is available in the nonlinear ODE case. Its principle consists in computing an equation (called the "IO equation") which is a consequence of the model equations and only depends on the model inputs, outputs and parameters. In the nonlinear ODE case, it is known since [8] that it can serve to decide the identifiability property of the model. It is known since [7] that it can also be used to determine a first guess of the parameters from experimental data. This first guess may then be refined by means of a nonlinear fitting algorithm (of type Levenberg-Marquardt) which requires many different numerical integrations of the model.

Designing analogue theories and algorithms in the IDE case is almost a completely open problem. The talk presents two contributions:

1. a symbolic method for computing an IO equation from a given nonlinear IDE model. This method is incomplete but it is likely to apply over an important class of models that are interesting for modelers. It relies on the elimination theory for differential algebra [4, 5] and on an algorithm for integrating differential fractions [3];

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2. an algorithm for the numerical integration of IDE systems, implemented within a new open source C library [1]. The library does not seem to have any available equivalent. Our algorithm is an explicit Runge-Kutta method which is restricted to Butcher tableaux specifically designed in order to avoid solving integral equations at each step.

Keywords: nonlinear integro-differential, input-output equation, parameter estimation, numerical integration

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Theorem
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Santiago de Compostela, June 18–22, 2018

Parametric b-functions for some hypergeometric ideals*

Francisco-Jesús Castro-Jiménez[1], Helena Cobo Pablos[1]

We denote by $D := \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ the Weyl algebra over the field $\mathbb{C}$.

The aim of this note is to study the b–function associated with a class of hypergeometric ideals $H_A(\beta) \subseteq D$ following [9, Section 5.1]. Let us recall the definition of $H_A(\beta)$. Given $A = (a_{ij})$ a $d \times n$ matrix of rank d with integer coefficients, we first consider the associated toric ideal $I_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \ldots, \partial_n]$

$$I_A := \mathbb{C}[\partial]\{\partial^u - \partial^v \mid u, v \in \mathbb{N}^n, Au = Av\}.$$

Moreover we consider the Euler operators, for $1 \leq i \leq d$

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n.$$

Then for any parameter vector $\beta \in \mathbb{C}^d$ the hypergeometric ideal is defined as

$$H_A(\beta) = D \cdot I_A + \sum_{1 \leq i \leq d} D(E_i - \beta_i).$$

Given a holonomic left ideal I in D and a nonzero weight vector $\omega \in \mathbb{R}^n$, we denote $in_{(-\omega,\omega)}(I) \subset D$ the initial ideal of I with respect to the filtration $(F_p)_{p \in \mathbb{R}}$ induced on D by the vector $(-\omega, \omega) \in \mathbb{R}^{2n}$. The $\mathbb{C}$–vector space F_p is defined as follows:

$$F_p := \mathbb{C}\{x^\alpha \partial^\beta \mid -\omega\alpha + \omega\beta \leq p\} \quad \text{for } p \in \mathbb{R}.$$

Kashiwara has introduced in (*On the Holonomic Systems of Linear Differential Equations, II.* Inventiones Math. 49, 121–135, 1978) the b–function $b_{I,\omega}(s)$ associated with the pair (I, ω), as the monic generator of the ideal

$$in_{(-\omega,\omega)}(I) \cap \mathbb{C}[s] \tag{1}$$

where $s := \sum_{i=1}^n \omega_i x_i \partial_i$. It is proven in *loc. cit. Theorem 2.7* that the ideal in (1) is nonzero. In this note we follow the presentation and notations of [9, §5] on this subject.

The polynomial $b_{I,\omega}(s)$ is called the b–function of the holonomic ideal $I \subset D$ with respect to the weight vector ω.

Previous b-functions are closely related to the classical notion of Bernstein polynomial (also called Bernstein-Sato polynomial) $b_f(s)$ associated with a given nonzero

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polynomial $f \in \mathbb{C}[x]$ (see e.g. [9, Lemma 5.3.11]). Bernstein polynomials have been introduced in [2] and [8] and represent fundamental invariants in singularity theory. There are several algorithms for computing Bernstein polynomials. Some of them are described in [5], [6], [4], and [1]. These and other algorithms have been implemented in the computer algebra systems `Asir`, `Macaulay2` and `Singular` among others. Nevertheless, in practice $b_f(s)$ is hard to compute even in the case of a polynomial f in two variables. In [3] the authors propose the algorithm `checkRoot` which, given a rational number α checks if it is a root of the Bernstein polynomial $b_f(s)$, and computes its multiplicity.

We simply denote $b_{\omega,\beta}(s) := b_{H_A(\beta),\omega}(s)$. We refer to [9] for the main results on hypergeometric ideals and the corresponding b–functions $b_{\omega,\beta}(s)$ *for generic parameters w and β* (see below for details). In [7] the authors describe bounds for the roots of $b_{\omega,\beta}(s)$.

In this paper we restrict ourselves to matrices of the form $A = (1, p, q)$ with integers $1 < p < q$ and p and q coprime. The first step is to describe the Gröbner fan of the toric ideal I_A, as defined in (T. Mora; L. Robbiano, The Gröbner fan of an ideal. *J. Symbolic Comput.* **6**(2-3) 183–208 (1988)) and in (B. Sturmfels, *Gröbner bases and convex polytopes*. University Lecture Series, 8. Providence RI, 1995.) We define a finite family of disjoint regions $R_i^{(k)} \subset \mathbb{R}^3$ which are the intersection of two half-spaces with the line $(1, p, q)\mathbb{R}$ in common (see Example). The possible integers k and i depend on the extended Euclidean division of q over p. We prove an equality $\mathbb{R}^3 = \bigcup_{i,k} \overline{R_i^{(k)}}$ such that for each $\omega \in R_i^{(k)}$, the initial ideal $in_\omega(I_A)$ is a monomial ideal and it is independent of ω.

In [9, Proposition 5.1.9.] there is a description of $b_{\omega,\beta}(s)$ for Zariski generic β and generic ω In (M.C. Fernández-Fernández, *Soluciones Gevrey de sistemas hipergeométricos asociados a una curva monomial lisa.* DEA, U. Sevilla, 2008.), the polynomial $b_{\omega,\beta}(s)$ is described for $\omega = (1, 0, 0)$ and β generic. Our main result is:

Theorem 0.1. *Given $R_i^{(k)}$, a facet of the Gröbner fan of I_A, there is a proper Zariski closed set $C_i^{(k)} \subset R_i^{(k)}$ such that if $\omega \in R_i^{(k)} \setminus C_i^{(k)}$ and β is generic the b–function is*

$$b_{\omega,\beta}(s) = \prod_{\alpha \in F_i^{(k)}} (s - \alpha)$$

for certain finite set $F_i^{(k)} \subseteq \mathbb{C}$. Moreover, if $\omega \in C_i^{(k)}$ or β is non-generic, the right hand side of previous equality gives a multiple of the b–function.

The set $F_i^{(k)}$ is explicitly described in terms of standard monomials of $in_{(-\omega,\omega)}(H_A(\beta))$. In the following example we sum up our results. Consider the matrix $A = (1, 3, 5)$. The Gröbner fan of $I_A \subset \mathbb{C}[\partial_x, \partial_y, \partial_z]$ consists of seven facets. Let us focus in one of them, namely $R_1^{(2)} = \{\omega \in \mathbb{R}^3 \mid 2\omega_1 + \omega_2 > \omega_3, \omega_1 + 3\omega_2 < 2\omega_3\}$. For any $\omega \in R_1^{(2)}$

$$in_\omega(I_A) = D\left(\partial_x^3, \partial_x^2\partial_y, \partial_x\partial_z, \partial_z^2\right).$$

Any complex number $\beta \neq 2$ is generic, and we have that

$$in_{(-\omega,\omega)}(H_A(\beta)) = D\left(\partial_x^2, \partial_x\partial_z, \partial_z^2, E-\beta\right).$$

We have $C_1^{(2)} = R_1^{(2)} \cap \{3\omega_1 + 4\omega_2 = 3\omega_3\}$. The b–function for $\omega \in R_1^{(2)} \setminus C_1^{(2)}$ and $\beta \neq 2$ is

$$b_{\omega,\beta}(s) = (s - \frac{\beta}{3}\omega_2)(s - \omega_1 - \frac{\beta-1}{3}\omega_2)(s - \frac{\beta-5}{3}\omega_2 - \omega_3).$$

If $\omega \in C_1^{(2)}$ and $\beta \neq 2$, the polynomial

$$(s - \frac{\beta}{3}\omega_2)(s - \omega_1 - \frac{\beta-1}{3}\omega_2)$$

is a multiple of the b–function. With `Singular` we check that in this case we obtain the true b–function and not just a multiple. If $\omega \in R_1^{(2)}$ but $\beta = 2$ we have the following multiple of the b–function:

$$\begin{cases} (s - \frac{2}{3}\omega_2)(s - \omega_1 - \frac{1}{3}\omega_2)(s - 2\omega_1)(s + \omega_2 - \omega_3) & \text{if } \omega \notin C_1^{(2)} \\ (s - \frac{2}{3}\omega_2)(s - \omega_1 - \frac{1}{3}\omega_2)(s - 2\omega_1) & \text{otherwise.} \end{cases}$$

Again, with `Singular` we check that this is indeed $b_{\omega,2}(s)$. However, if we consider the region $R_2^{(2)} = \{\omega \in \mathbb{R}^3 \mid \omega_1 + 3\omega_2 > 2\omega_3, 3\omega_3 > 5\omega_2\}$, we have $\beta = 1, 2, 4, 7$ as non-generic values, and for $\omega \in R_2^{(2)}$ and $\beta = 2$ we give a polynomial with five roots, and only four of them are the roots of $b_{\omega,2}(s)$. If $\omega \in \mathbb{R}^3 \setminus \bigcup_{i,k} R_i^{(k)}$ the study of $b_{\omega,\beta}(s)$ is a work in progress.

Keywords: b–function, hypergeometric ideal.

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Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

Reduction operators and completion of linear rewriting systems

Cyrille Chenavier[1]

In rewriting theory, the confluence property guarantees the coherence of calculi. In this talk, we study the confluence property for linear rewriting systems defined by reduction operators. We use this approach to provide a lattice description of obstructions to confluence. We deduce lattice formulations of the completion procedure as well as a method for extending linear rewriting systems so that they become confluent.

Keywords: Reduction operators, Lattice structure, Confluence, Completion procedure

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Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

Observability and orders of derivatives of data

Sette Diop[1]

Observability of nonlinear systems has been approached using differential algebraic geometry with quite interesting breakthroughs in this systems theory notion. Among detailed aspects to be studied is the relationship between observability of, say z, and the minimum order of derivatives of data. This relationship is an ingredient in the design and the complexity of observers. This talk will give new insights in this topic.

Keywords: Observability, systems theory, differential algebra

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Santiago de Compostela, June 18–22, 2018

Effective criterion to test differential transcendence of special functions.

Carlos Arreche[1], **Thomas Dreyfus**[2], **Julien Roques**[3]

Consider a field $\mathbf{k}$ equipped with an automorphism ϕ. Typical examples are

- $\mathbf{k} = \mathbb{C}^{\mathbb{Z}}$, $\phi(u_n) := (u_{n+1})$;
- $\mathbf{k} = \mathbb{C}(x)$, $\phi f(x) := f(x+1)$;
- $\mathbf{k} = \mathbb{C}(x)$, $\phi f(x) := f(qx)$, $q \in \mathbb{C}^*$;
- $\mathbf{k} = \cup_{\ell \in \mathbb{N}^*} \mathbb{C}(x^{1/\ell})$, $\phi f(x) := f(x^p)$, $p \in \mathbb{N}^*$.

A difference equation is a linear equation of the form

$$a_0 y + \cdots + a_n \phi^n(y) = 0,$$

with $a_0, \ldots, a_n \in \mathbf{k}$. The difference Galois theory, see [1], attaches to such equation a linear algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ that measures the algebraic relations among the solutions of the difference equation. More recently, it has been developed in [2] a Galois theory that aims at understanding the algebraic and differential relations among the solutions of the difference equation

The goal of this talk is to give explicit and computable criterias to ensure that a solutions of an order two difference equation does not satisfy any algebraic differential equations in coefficients in $\mathbf{k}$. We apply this criterion to the elliptic analogue of the hypergeometric functions.

Keywords: Difference Galois theory

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Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

Symbolic computation for integro-differential-time-delay operators with matrix coefficients

Thomas Cluzeau[1], **Jamal Hossein Poor**[2], **Alban Quadrat**[3], **Clemens G. Raab**[4], **Georg Regensburger**[5]

In order to facilitate symbolic computations with systems of linear functional equations, we require an algebraic framework for such systems which enables effective computations in corresponding rings of operators. We briefly explain the recent developed tensor approach from scalar equations [1] to the matrix case [2], by allowing noncommutative coefficients. Noncommutative coefficients even allow to handle systems of generic size. Normal forms are a key ingredient for computing with operators and rely on a confluent reduction system.

The tensor approach is flexible enough to cover many operators, like integral operators, that do not fit the well established framework of skew-polynomials. For instance, it can be used to construct the ring of integro-differential operators with linear substitutions (IDOLS) having (noncommutative) matrix coefficients, containing the ring of integro-differential-time-delay operators. In the `Mathematica` package `TenRes` we provide support for tensor reduction systems [3]. In addition, we implement the ring of IDOLS and corresponding normal forms. We illustrate how, by elementary computations in this framework, results like the method of steps can be found and proven in an automated way. We also apply normal forms of IDOLS to partly automatize certain computations related to differential time-delay systems, e.g. Artstein's transformation [4] and its generalization [5].

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Keywords: integro-differential operators with linear substitutions, Artstein's reduction, algebraic analysis approach to linear systems theory

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Santiago de Compostela, June 18–22, 2018

Low-Order Recombinations of C-Finite Sequences

Maximilian Jaroschek[1,2], Manuel Kauers[1], Laura Kovács[2]

One of the central open problems for C-finite sequences, that is sequences that admit a linear recurrence equation with constant coefficients, is the Skolem problem, which asks if a given sequence includes the term 0. Special instances for which an answer can be given algorithmically include the case where there exists an annihilating recurrence of order less than or equal to 4. The Skolem problem is of particular interest in program verification, as the values of loop variables in practice often describe C-finite sequences. We investigate how to combine these C-finite sequences via term-wise multiplication and addition so that the resulting sequences admit recurrences of low order. These combinations then can be used as inequality loop invariants in automatic program analysis.

Keywords: C-finite Sequences, Skolem Problem, Invariant Generation

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Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

Some Properties and Invariants of Multivariate Difference-Differential Dimension Polynomials

Alexander Levin[1]

Multivariate dimension polynomials associated with finitely generated differential and difference field extensions arise as natural generalizations of the univariate differential and difference dimension polynomials defined respectively in [1] and [2]. It turns out, however, that they carry more information about the corresponding extensions than their univariate counterparts (see [3, Theorem 4.2.17] and [4]). In this presentation we extend the known results on multivariate dimension polynomials to the case of difference-differential field extensions with arbitrary partitions of sets of basic operators. We also describe some properties of multivariate dimension polynomials and their invariants. The following is the outline of the talk.

Let K be a difference-differential field, $\operatorname{Char} K = 0$, and let $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ be basic sets of derivations and automorphisms of K, respectively. Below we often use the prefix Δ-σ- instead of "difference-differential". Suppose that the sets Δ and σ are represented as unions of disjoint subsets: $\Delta = \cup_{i=1}^{p} \Delta_i$ and $\sigma = \cup_{j=1}^{q} \sigma_j$ where $\operatorname{Card} \Delta_i = m_i$ $(1 \le i \le p)$ and $\operatorname{Card} \sigma_i = n_i$ $(1 \le i \le q)$. Let Λ denote the free commutative semigroup of all power products of the form $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ where $k_\mu \in \mathbb{N}$, $l_\nu \in \mathbb{Z}$ and for every such λ, let

$$\operatorname{ord}_{\Delta_i} \lambda = \sum_{\mu \in \Delta_i} k_\mu \ \text{ and } \ \operatorname{ord}_{\sigma_j} \lambda = \sum_{\nu \in \sigma_j} |l_\nu|$$

$(1 \le i \le p,\ 1 \le j \le q)$. Furthermore, for any $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$, let $\Lambda(r_1, \dots, r_{p+q}) = \{\lambda \in \Lambda \,|\operatorname{ord}_{\Delta_i} \lambda \le r_i$ for $i = 1, \dots, p$ and $\operatorname{ord}_{\sigma_j} \lambda \le r_{p+j}$ for $j = 1, \dots, q\}$. The following theorem generalizes the main result of [4].

Theorem 0.2. *Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta \in \mathbb{Q}[t_1, \dots, t_{p+q}]$ (called the Δ-σ-dimension polynomial of the extension L/K) such that*

(i) $\Phi_\eta(r_1, \dots, r_{p+q}) = \operatorname{tr.deg}_K K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+q})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$ (it means that there exist $s_1, \dots, s_{p+q} \in \mathbb{N}$ such that the equality holds for all $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$ with $r_1 \ge s_1, \dots, r_{p+q} \ge s_{p+q}$);

(ii) $\deg_{t_i} \Phi_\eta \leq m_i$ *(*$1 \leq i \leq p$*)*, $\deg_{t_{p+j}} \Phi_\eta \leq n_j$ *(*$1 \leq j \leq q$*) and* $\Phi_\eta(t_1, \dots, t_{p+q})$ *can be represented as*

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n_1} \dots \sum_{i_{p+q}=0}^{n_q} a_{i_1 \dots i_{p+q}} \binom{t_1 + i_1}{i_1} \dots \binom{t_{p+q} + i_{p+q}}{i_{p+q}}$$

where $a_{i_1 \dots i_{p+q}} \in \mathbb{Z}$ *and* $2^n \mid a_{m_1 \dots m_p n_1 \dots n_q}$.

We sketch the proof of this theorem and present a method of computation of the polynomial Φ_η based on a generalization of the Ritt-Kolchin method of characteristic sets. Furthermore, we determine invariants of a Δ-σ-dimension polynomial, i. e., numerical characteristics of the Δ-σ-field extension that are carried by such a polynomial and that do not depend on the set of Δ-σ-generators this Δ-σ-dimension polynomial is associated with. We also give conditions under which the Δ-σ-dimension polynomial is of the simplest possible form.

Keywords: Difference-differential field extension, Dimension polynomial, Characteristic set

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Computer algebra and the Lanczos problems in arbitrary dimension

J.-F. Pommaret[1]

When $\mathcal{D}$ is a linear partial differential operator of any order, a direct problem is to look for an operator $\mathcal{D}_1$ generating the compatibility conditions (CC) $\mathcal{D}_1\eta = 0$ of $\mathcal{D}\xi = \eta$. We may thus construct a differential sequence with successive operators $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2, ...$, where each operator is generating the CC of the previous one. Introducing the formal adjoint $ad()$, we have $\mathcal{D}_i \circ \mathcal{D}_{i-1} = 0 \Rightarrow ad(\mathcal{D}_{i-1}) \circ ad(\mathcal{D}_i) = 0$ but $ad(\mathcal{D}_{i-1})$ may not generate all the CC of $ad(\mathcal{D}_i)$. When $D = K[d_1, ..., d_n] = K[d]$ is the (non-commutative) ring of differential operators with coefficients in a differential field K, it gives rise by residue to a differential module M over D. The homological extension modules $ext^i(M) = ext^i_D(M, D)$ with $ext^0(M) = hom_D(M, D)$ only depend on M and are measuring the above gaps, independently of the previous differential sequence.

The purpose of this talk is to explain how to compute extension modules for certain Lie operators involved in the formal theory of Lie pseudogroups in arbitrary dimension n. In particular, we prove that the extension modules highly depend on the Vessiot structure constants c. When one is dealing with a Lie group of transformations or, equivalently, when $\mathcal{D}$ is a Lie operator of finite type, then we shall prove that $ext^i(M) = 0, \forall 0 \leq i \leq n-1$. It will follow that the Riemann-Lanczos and Weyl-Lanczos problems just amount to prove such a result for $i = 2$ and arbitrary n when $\mathcal{D}$ is the Killing or conformal Killing operator. We finally prove that $ext^i(M) = 0, \forall i \geq 1$ for the Lie operator of infinitesimal contact transformations with arbitrary $n = 2p + 1$. Most of these new results have been checked by means of computer algebra.

Keywords: Differential sequence, Variational calculus, Differential constraint, Control theory, Killing operator, Riemann tensor, Bianchi identity, Weyl tensor, Lanczos tensor, Contact transformations, Vessiot structure equations

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Algebraic proofs of operator identities

Jamal Hossein Poor[1], **Clemens G. Raab**[2], **Georg Regensburger**[2]

Many interesting properties of linear operators can be phrased as operator identities, which then can be proven algebraically. In practice, however, linear operators often map between different spaces, then we can no longer add or compose any two such operators. For instance, this already happens with rectangular matrices or with differential operators having rectangular matrix coefficients.

In order to still be able to do meaningful symbolic computations with such operators on the computer, an algebraic framework is needed that deals with the corresponding domains and codomains of operators when adding and multiplying operators. In principle, symbolic computation with such operators (or matrices) would require at each step taking care of the domains and codomains of those operators (or of the formats of the matrices). In contrast, we aim at an a-posteriori justification of an identity, independent of how it was computed algebraically.

In this talk we present first results towards such an algebraic framework based on quivers and noncommutative Gröbner bases, which could be applied to operators with rectangular matrix coefficients. We will also present examples from the theory of generalized inverses using noncommutative Gröbner bases.

Keywords: Linear operators, noncommutative Gröbner bases

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Applications of Computer Algebra – ACA2018
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Definite Integration of D-finite Functions via Generalized Hermite Reduction

Alin Bostan[1] **Frédéric Chyzak**[1], **Pierre Lairez**[1], **Bruno Salvy**[2]

Hermite reduction is a classical algorithmic tool in symbolic integration. It is used to decompose a given rational function as a sum of a function with simple poles and the derivative of another rational function. It provides a canonical form modulo derivatives of rational functions. We extend Hermite reduction to arbitrary linear differential operators instead of the pure derivative, and develop efficient algorithms for this reduction. We then apply the generalized Hermite reduction to the computation of linear operators satisfied by definite integrals. The resulting algorithm is a generalization of reduction-based methods for creative telescoping.

Keywords: Hermite reduction, symbolic integration, creative telescoping

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Solution of non homogenous Ordinary Differential Equations using Parametric Integral Method

Thierry N. Dana-Picard[1], **David G. Zeitoun**[2]

The solution of non homogenous ordinary differential equation (ODE) is an important research subject appearing in numerous engineering fields. When the ODE is associated with boundary conditions (BC), the problem is referred to as a Boundary Value Problem (BVP). Numerical schemes such as finite differences and finite elements have been used for the solution of such problem.

A general homogeneous ODE may be expressed as:

$$\begin{cases} \sum\limits_{n=0}^{n=p} a_n(x)\frac{d^{(n)}y}{dx^n} = 0 \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{1}$$

This equation may be decomposed into the homogenous part and a non homogenous part, using a MacLaurin expansion of each coefficient $a_n(x)$. For any $n \in \{1, ..., p\}$, we have:

$$a_n(x) = a_n(0) + a_n'(0)x + \frac{x^2}{2}a_n''(0) + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}a^{(n)}(0) \tag{2}$$

Inserting this last identity into Equation (1) leads to:

$$\begin{cases} L_0(y) = -L(y) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{3}$$

where the differential operator L is defined by:

$$L = \sum_{n=0}^{n=p} \left[\sum_{n=1}^{\infty} \frac{x^n}{n!}a^{(n)}(0) \right] \frac{d^{(n)}}{dx^n} \tag{4}$$

The operator L_0 is defined as :

$$L_0 = \sum_{n=0}^{n=p} a_n(0)\frac{d^{(n)}}{dx^n} \tag{5}$$

In this contribution we present a general methodology based on the Adomian decomposition method (ADM) as described in [3]), where the inverse operator L^{-1} is expressed in terms of eigenvectors and eigenvalues expansion. The ADM is a systematic method for solution of either linear or nonlinear operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It enables to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) (see [5]) without physical restrictive assumptions, such as those required by linearization, perturbation, ad hoc assumptions, and guessing the initial term or a set of basis functions.

Using ADM, we denote a possible solution by $y(x) = \sum\limits_{m=0}^{\infty} y_m(x)$. A general solution of the non homogenous ODE may be found in an iterative way as follows:

- Solve for $y_0(x)$:

$$\begin{cases} L_0(y_0) = 0 \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{6}$$

- Solve for $y_m(x); m = 1, 2, ...$

$$\begin{cases} L_0(y_m) = -L(y_{m-1}) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{7}$$

After solving for $y_0(x)$, the general solution for Equations (7) may be derived using the Green function associated with the operator L_0.

$$\begin{cases} L_0(G(x,\xi)) = \delta(x - \xi) \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases}$$

Using Equation (8) and suitable boundary conditions for $G(x, \xi)$, we obtain an iterative solution for $m \geq 1$:

$$y_m(x) = \int_a^b G(x,\xi) L(y_{m-1}(\xi) d\xi \tag{8}$$

In a large class of boundary value problems, the Green function $G(x, \xi)$ may be expressed as an eigenfunction expansion as follows:

$$G(x,\xi) = \sum_{r=1}^{r=q} \frac{\phi_r(x)\phi_r(\xi)}{\lambda_r} \tag{9}$$

where λ_r is the eigenvalue associated with the eigenfunction $\phi_r(x)$ which is the solution of the following ODE:

$$\begin{cases} L_0(\phi_r) = \lambda_r \phi_r \\ a \leq x \leq b \\ (BC) \quad \text{at } x = a \quad \text{and at } x = b \end{cases} \tag{10}$$

So finally the iterative Adomian solution of Equation (7) may be written as:

$$y_m(x) = \sum_{r=1}^{r=q} \frac{\phi_r(x)}{\lambda_r} \int_a^b \phi_r(\xi) L(y_{m-1}(\xi) d\xi \tag{11}$$

In this talk, this last expression will be used to generate different types of iterative algorithms for the solution of the BVP. This iterative algorithm generates an iterative algorithm which can be implemented in a CAS. As examples, we will present solutions of groundwater flow through non homogenous formations using parametric integral solutions. This type of integrals have been already analysed by the authors in [1, 2, 4].

Keywords: parametric integral, non homogenous ODE, Adomian decomposition method

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Desingularization in the q-Weyl algebra

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The desingularization problem has been primarily studied for linear differential operators with polynomial coefficients. The solutions of such an equation are called *D-finite* functions. It is well known that a singularity at a certain point x_0 of one of the solutions must be reflected by the vanishing (at x_0) of the leading coefficient of the differential equations. However, the converse however is not always true: not every zero of the leading coefficient polynomial induces a singularity of at least one function in the solution space. The purpose of desingularization is to construct another equation, whose solution space contains that of the original equation, and whose leading coefficient vanishes only at the singularities of the previous solutions. Typically, such a desingularized equation will have a higher order, but a lower degree for its leading coefficient. In summary, desingularization provides some information about the solutions of a given differential equation.

The authors of [1, 3] give general algorithms for the Ore case. However, from a theoretical point of view, the story is not yet finished, in the sense that there is no order bound for desingularized operators in the Ore case. We consider the desingularization problem in the first q-Weyl algebra. Our main contribution is to give an order bound for desingularized operators, and thus derive an algorithm for computing desingularized operators in the first q-Weyl algebra. In addition, an algorithm is presented for computing a generating set of the first q-Weyl closure of a given q-difference operator. As an application, we certify that several instances of the colored Jones polynomial from knot theory are Laurent polynomial sequences by computing the corresponding desingularized operator.

Keywords: Desingularization, q-Weyl algebra, Knot Theory

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