SPECIAL SESIONS

Applications of Computer Algebra - ACA2018

June 18–22, 2018

Santiago de Compostela, Spain

S8
Dynamic Geometry and Mathematics Education

Wednesday

Wed 20th, 10:00 - 10:30, Aula 10 – Thierry Dana-Picard:
A new approach to automated study of isoptic curves

Wed 20th, 10:30 - 11:00, Aula 10 – Roman Hašek:
Exploration of dual curves using dynamic geometry and computer algebra system

Wed 20th, 11:30 - 12:00, Aula 10 – Setsuo Takato:
Programming in KeTCindy with Combined Use of Cinderella and Maxima

Wed 20th, 12:00 - 12:30, Aula 10 – Raúl M. Falcón:
Discovering properties of bar linkage mechanisms based on partial Latin squares by means of Dynamic Geometry Systems

Wed 20th, 12:30 - 13:00, Aula 10 – Philippe R. Richard:
Issues and challenges about instrumental proof

Wed 20th, 13:00 - 13:30, Aula 10 – Round table:
Dynamic Geometry and Computer Algebra Systems in Mathematics instruction

Organizers

Tomás Recio
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Aim and cope

Dynamic geometry environments (DGE) have emerged in the last half-century with an ever-increasing impact in mathematics education. DGE enlarges the field of geometric objects subject to formal reasoning, for instance, simultaneous operations with many geometric objects. Today DGE open the possibility of investigating visually and formulating conjectures, comparing objects, discovering or proving rigorously properties over geometric constructions, and Euclidean elementary geometry is required to reason about them.

Along these decades various utilities have been added to these environments, such as the manipulation of algebraic equations of geometric objects or the automated proving and discovering, based on computer algebra algorithms, of elementary geometry statements. Moreover, some intelligent tutoring systems for Euclidean geometry based in DGE have been developed.

The merging of these tools (DGE, automated proving and intelligent tutoring systems) is, thus, a very natural, challenging and promising issue, currently involving logic, symbolic computation, software development, algebraic geometry and mathematics education experts all from over the world.

The Special Session intends to be a forum for:

- presenting the current state of the art concerning the design and implementation of automatic reasoning features on dynamic geometry systems and intelligent tutoring systems;
- fostering a debate concerning the role and use of such features in mathematics education, in general, and their potential impact in proof and proving conception in the classroom, in particular.

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A new approach to automated study of isoptic curves

Thierry Dana-Picard[1], Zoltan Kovács[2]

Let $\mathcal{C}$ be a plane curve. For a given angle θ with $0 \leq \theta \leq 180°$), a θ-*isoptic* of $\mathcal{C}$ is the geometric locus of points in the plane through which pass a pair of tangents with an angle of θ between them. The special case for which $\theta = 90°$ is called an *orthoptic curve*. The orthoptics of conics are well known: the directrix of a parabola, the director circle of an ellipse, and the director circle of a hyperbola (in this case, its existence depends on the eccentricity of the hyperbola).

Orthoptics and θ-isoptics can be studied for other curves, in particular for closed smooth convex curves; see [1]. Isoptics of an astroid are studied in [2] (see Figure 1) and of Fermat curves in [3]. If $\mathcal{C}$ is an astroid, there exist points through which pass 3 tangents to $\mathcal{C}$, and two of them are perpendicular. These works combine geometrical experimentation with a Dynamical Geometry System (DGS) GeoGebra and algebraic computations with a Computer Algebra System (CAS). For them, the curve has been defined by a parametrization. A new approach to these curves is proposed, using

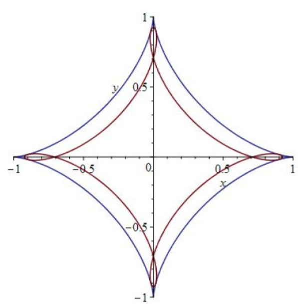

Figure 1: The 45-isoptic of the astroid

the DGS GeoGebra, not only its geometrical part but also its CAS component. The central feature is the connection between the two components of the same software package, enabling automatic switching between different registers of representation. This approach enables to determine the θ-isoptics of various curves, either closed or not. Moreover, the dynamics of the work is essential for the study of the convexity of the θ-isoptic. Students, teachers and researchers can make their own experiments, checking the existence of flexes, changing curves to look for invariant properties, etc.

We demonstrate this approach with GeoGebra applets [4] and [5] for parabolas and other planes curves, either closed or not. Here there is no need to use parametric equations for defining $\mathcal{C}$, and the work is based on implicit equations.

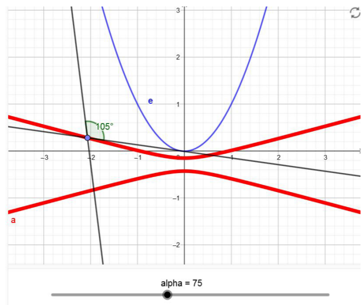

Figure 2: A screenshot of an applet

This automated work allows undergraduates to be acquainted with an advanced topic in Differential Geometry.

Keywords: Plane curves, isoptics, automated proof, dynamical geometry

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Discovering properties of bar linkage mechanisms based on partial Latin squares by means of Dynamic Geometry Systems

Raúl M. Falcón[1]

Dynamic Geometry Systems (DGSs) have recently been proposed in mechanical engineering as an alternative to deal with the teach, design, analysis and implementation of mechanisms [1, 6, 7, 8, 9]. Recall that a *mechanism* is any set of rigid bodies connected by *joints* so that force and motion are transmitted among themselves. A *link* within a mechanism is any of its rigid bodies having at least two different joints. A *bar linkage mechanism* is any mechanism in which all its rigid bodies are bars and at least one of them is a link. The study of the relative motion that occurs between each pair of connected bars within one such a mechanism enables its characterization. In this regard, the *degree of freedom of a joint* connecting two bars is defined as the number of independent parameters that are required to determine the relative position of one of the bars with respect to the other one. This has influence on the different *coupler curves* that are generated by the joints within each bar. The study and analysis of such curves enable one to design optimal devices and give rise, therefore, to important applications in Technology and Engineering. Since coupler curves can be described as loci of points satisfying certain geometrical constraints derived from the lengths and connections of bars within a mechanism, DGSs constitute an interesting tool to investigate and characterize them from a visual and dynamical point of view.

In this work, we focus on the use of DGSs to deal with those bar linkage mechanisms such that the distance matrix defined by their joints constitutes a unipotent partial Latin square satisfying certain conditions. Recall that a *partial Latin square* of order n is an $n \times n$ array in which each cell is either empty or contains an element of a finite set of n symbols so that each symbol occurs at most once in each row and in each column. Let $\mathrm{PLS}(n)$ denote the set of partial Latin squares of order n having $[n] := \{0, 1, \ldots, n-1\}$ as set of symbols. The rows and columns of every array in $\mathrm{PLS}(n)$ are supposed to be naturally indexed by the elements of the set $[n]$. Throughout our study, we focus on the subset of partial Latin squares $L = (l_{ij}) \in \mathrm{PLS}(n)$ that are also

i. *reduced*, that is, such that $l_{0i} = i$ and $l_{j0} = j$, for all $i, j \in [n]$ satisfying that the cells $(0, i)$ and $(j, 0)$ in L are non-empty;

ii. *zero-diagonal*, that is, $l_{ii} = 0$, for all $i \in [n]$; and

iii. *symmetric*, that is, $l_{ij} = l_{ji}$, for all $i, j \in [n]$.

To avoid degeneracy and disjoint unions of disconnected mechanisms, we also suppose that

iv. every row and every column of L must contain at least one symbol of the set $[n] \setminus \{0\}$;

v. for each pair $(i,j) \in [n] \times [n]$ such that $l_{ij} \in [n]$, there exists a positive integer $k \in [n]$ such that either $l_{kj} \in [n]$ or $l_{ik} \in [n]$. This involves every bar in the mechanism to be connected to at least one other bar by a joint.

Finally, in order to get linkage mechanisms, the following condition is also required:

vi. If every symbol in $[n] \setminus \{0\}$ appears exactly twice in L, then they cannot be all of them in a same row and column of L.

Let $\mathcal{M}_n$ denote the set of partial Latin squares of order n satisfying Conditions (i)–(vi). This set is preserved by isomorphisms. Recall that two partial Latin squares $L = (l_{ij})$ and $L' = (l'_{ij})$ in $\mathrm{PLS}(n)$ are *isomorphic* if there exists a permutation π on the set $[n]$ such that $\pi(l_{ij}) = l'_{\pi(i)\pi(j)}$, for all $i, j \in [n]$ such that $l_{ij} \in [n]$. To be isomorphic constitutes an equivalence relation among partial Latin squares. The distribution of partial Latin squares into isomorphism classes is known [2, 3, 5], for order $n \leq 6$.

Let $M(L)$ denote the set of bar linkage mechanisms that are associated to a given partial Latin square $L = (l_{ij}) \in \mathcal{M}_n$ as follows:

1. There exists a bar B_{ij} within the mechanism, for each pair $(i,j) \in [n] \times [n]$ such that $i < j$ and $l_{ij} \in [n] \setminus \{0\}$.

2. Two different bars B_{ij} and $B_{i'j'}$ within the mechanism are connected by a joint J_k if and only $\{i,j\} \cap \{i',j'\} = \{k\} \neq \emptyset$. This joint is placed in the corresponding extreme of each bar.

3. Two different bars B_{ij} and $B_{i'j'}$ within the mechanism have the same length if and only if $l_{ij} = l_{i'j'}$.

DGSs constitutes an interesting tool to deal with the study, analysis and characterization of the bar linkage mechanisms in the set $M(L)$. To this end, we consider each symbol $k \in [n] \setminus \{0\}$ to be uniquely associated to a slider s_k so that the length of each bar B_{ij} such that $l_{ij} = k$ is the value given by such a slider s_k (see Figure 1).

In this work, we make use of the mentioned sliders to teach, investigate properties and formulate conjectures about lengths of bars and coupler curves related to those mechanisms associated to partial Latin squares in the set $\mathcal{M}_n$, according to their distribution into isomorphism classes. In this regard, remark the recent study [4] about loci of points whose distance matrix constitutes a partial Latin square satisfying Conditions (i)–(iii). Further, the inclusion on new sliders within each worksheet under consideration enables us to deal with different parameters that characterize our bar

0	1	2		3
1	0		2	
2		0	4	
	2	4	0	
3				0

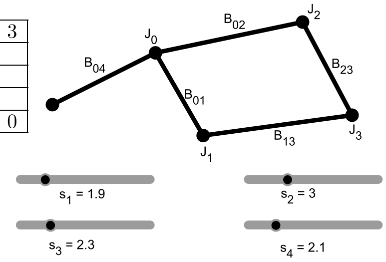

Figure 1: Dynamical study of a bar linkage mechanism based on a partial Latin square.

linkage mechanisms, as the degree of freedom, the transmission ratio, or the mechanical advantage, amongst others. All the constructions that have been developed in this work are available online in the official repository of GEOGEBRA, at the address `https://www.geogebra.org/m/crvJ7CzX`.

Keywords: Linkage systems, dynamic geometry, partial Latin square.

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Exploration of dual curves using dynamic geometry and computer algebra system

Roman Hašek[1]

This submission deals with the use of the dynamic mathematics software GeoGebra to determine the dual curve to the given curve and inspect its properties. The combination of dynamic geometry tools with computer algebra functions allows a user to take both geometric and algebraic perspectives on this issue. The dual curve to an algebraic curve is a curve born from the duality between points and lines in the projective plane. Writing the equation of a curve laying in this plane in homogeneous coordinates $[x_0, x_1, x_2]$ its tangents can be taken as points in the dual plane written in the coordinates $[y_0, y_1, y_2]$. Then the locus of these points is the dual curve to the given curve, [2].

We will show both the geometric model of the dual curve and the algebraic derivation of its equation in the talk. The geometric approach to display the dual curve shape is based on the polar reciprocity which is realized through the inversion in a circle here, [3, 4], see Fig. 1.

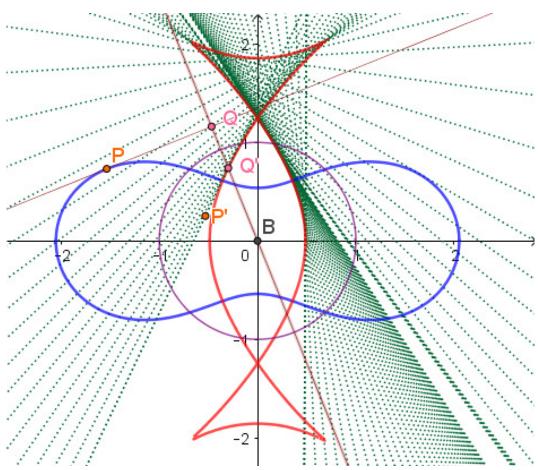

Figure 1: Dual curve to the Cassini oval as an envelope of lines dual to the points of the oval

The algebraic derivation of the dual curve equation is based on the idea that the related polynomial in indeterminates y_0, y_1, y_2 is a component of the Gröbner basis of the ideal of polynomials describing the aforesaid act of transition from a tangent line of the curve in a projective space to the point in its dual space, [5]. For example, considering the astroid with the Cartesian equation

$$27x^2y^2 + (x^2 + y^2 - 1)^3 = 0, \tag{1}$$

written in homogeneous coordinates $[x_0, x_1, x_2]$ as

$$h = x_0^6+x_1^6-x_2^6+3x_0^2x_1^4+3x_0^2x_2^4+3x_0^4x_1^2-3x_0^4x_2^2+3x_1^2x_2^4-3x_1^4x_2^2+21x_0^2x_1^2x_2^2 = 0, \tag{2}$$

the polynomial defining its dual is such a member of the Gröbner basis of the ideal of polynomials in indeterminates $x_0, x_1, x_2, y_0, y_1, y_2$

$$I = \langle y_0 - h'_{x_0}, y_1 - h'_{x_1}, y_2 - h'_{x_2}, h\rangle \tag{3}$$

that contains only indeterminates y_0, y_1, y_2. Its existence follows from the Elimination theorem, [1]. To derive the equation in GeoGebra we use the `Eliminate` command and, after transformation into the Cartesian equation

$$x^2y^2 - x^2 - y^2 = 0, \tag{4}$$

we can display the dual curve as shown in Fig. 2.

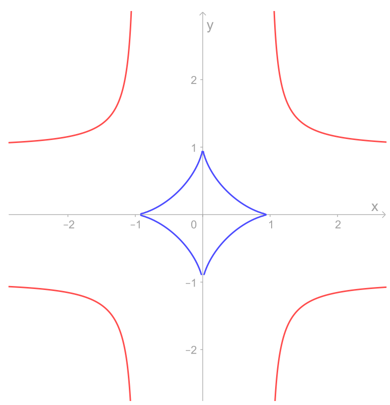

Figure 2: Dual curve (red) to the astroid (blue)

Apart from modeling the dual curve and the derivation of its equation we will also focus on the educational potential of this topic in the talk. The history of the notion of the dual curve is inter alia associated with the story of "the duality paradox" [3], which is worth mentioning when the concept of duality of projective space is taught. Moreover, the relation between a curve and its dual reveals concrete examples of how the duality works, [4]. See for example the correspondence between points and lines belonging to the dual curves in Figure 3, namely the correspondence between bitangents of the oval and nodes of its dual curve or the correspondence between inflexion points of the former and the cusps, more precisely tangents in them, of the latter. The utilization of dynamic geometry to explore these situations will also be presented through several particular examples.

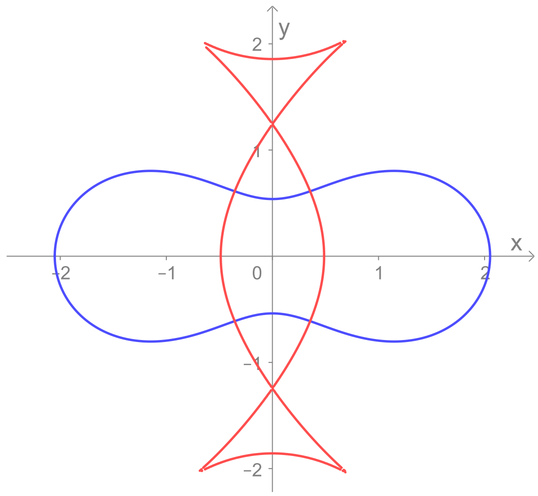

Figure 3: The Cassini oval (blue) and its dusl curve (red)

Keywords: Computer algebra, dual curve, dynamic geometry, Gröbner basis

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Issues and challenges about instrumental proof

Philippe R. Richard[1], Fabienne Venant[2], Michel Gagnon[3]

The notion of instrumental proof is relatively new. If the term is little used in didactic literature, its natural association with technologies, old and new, seems self-evident.

On the epistemological side, the discovery of Archimedes's palimpsest recently allowed us to better understand how the weighing method was a kind of mechanical proof, which suggests to the point that the association between proof and artifacts/tools is rather old. Similarly, computer proofs such as those of the four-colour theorem –first shown in 1976 by Kenneth Appel and Wolfgang Haken, then formally addressed in 2005 using Coq software by Georges Gonthier and Benjamin Werner– offer proofs where they are algorithms that base the decision or the verification of all cases, reflecting an unavoidable reality of contemporary mathematical work. Whether they are physical or logical, the use of tools in a validation situation certainly renews the usual idea that we have between the concepts of proof, modelling and representation of knowledge.

On the didactic side, there seems to be a constant struggle with paradoxes. The student is asked to prove propositions, but he or she now has an automated reasoning tool. It requires him or her to work with meaningful knowledge and to transform it, but by working more and more at the interface of computer tools that manage both a part of the representation and treatment, and often even experimenting on mathematical objects (e.g. dynamic figures) as a physicist does with objects of his own domain. And all this, without the teacher can refer to mathematics that could be described as technological, since he was initiated to a deductive science that has developed traditionally in writing.

It is then by extending ideas that we have already exposed in our work, including the recent paper [3], *The Concept of Proof in the Light of Mathematical Work*, and resuming conclusions of our current research projects (design of the tutorial system QED-Tutrix in high school geometry [1], the use of the Automated Reasoning Tools (ART) [2] in teacher training) that we approach the question of instrumental proofs. With this attitude, the subject-milieu interaction is a unit of epistemic necessity, the subject can be both a reader, to consider traditional proofs, and the user of software or a mathematical machine. The notions of reasoning in action and reasoning that unfold differently than with the discourse will be treated, as well as the theory of mathematical working spaces in which the question of the coordination of discursive, semiotic and instrumental geneses arise between an epistemological and a cognitive plan.

Keywords: Instrumental proof, mathematical working space, instrumented reasoning, algorithmic, physics

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Applications of Computer Algebra – ACA2018
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Programming in KeTCindy with Combined Use of Cinderella and Maxima

S. Takato[1], S. Yamashita[2] J.A. Vallejo[1]

Printed materials are often distributed to the classes at the college level. For such materials, line drawing type figures are more suitable. KeTCindy, a macro package of CindyScript which is a programming language implemented in Cinderella, supports line drawing of 3D figures. To produce these 3D figures with KeTCindy, it is fundamentally important to find intersections of projection curves. The combined use of KeTCindy, Cinderella, and Maxima is an effective tool to develop such programs.

Keywords: KeTCindy, Cinderella, Maxima

Mathematics teachers at the college level often distribute printed materials to their alumni. For such materials, figures presented as line drawings are better suited, because students can write their own remarks over them on the paper. KETCindy, a macro package of CindyScript (which is a programming language implemented in Cinderella), can produce fine figures for LATEX. Furthermore, KETCindy supports line drawing of 3D figures as explained below.

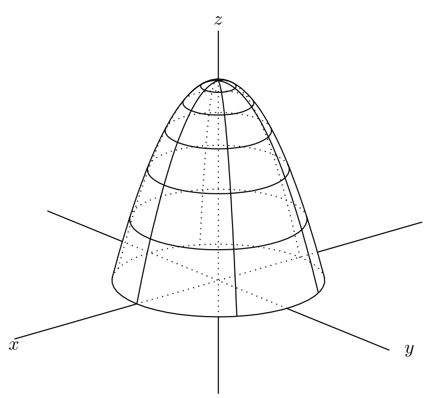

Fig.0

To produce these 3D figures, KETCindy follows the following steps:

1. To find silhouette lines of the surface, in the figure above those are given by $x = u\cos v,\ y = u\sin v,\ z = 4 - u^2$. Data are obtained from an implicit function of the form

$$J(u,v) = \frac{dX}{du}\frac{dY}{dv} - \frac{dX}{dv}\frac{dY}{du} = 0\,,$$

where $(X, Y) = \mathrm{Proj}(x, y, z)$ is the map to the plane of projection.

2. To find the intersections of silhouette lines and a projection curve.

3. To divide the curve by these intersects, and to decide whether each separation is hidden by the surface or not.

Of the above, the second item is of fundamental importance, but it represents a difficult task in the case of contacting curves because curves are numerically polygonal lines. The following figures demonstrate this setting: The right panel shows an enlarged figure at a contact point presented on the left.

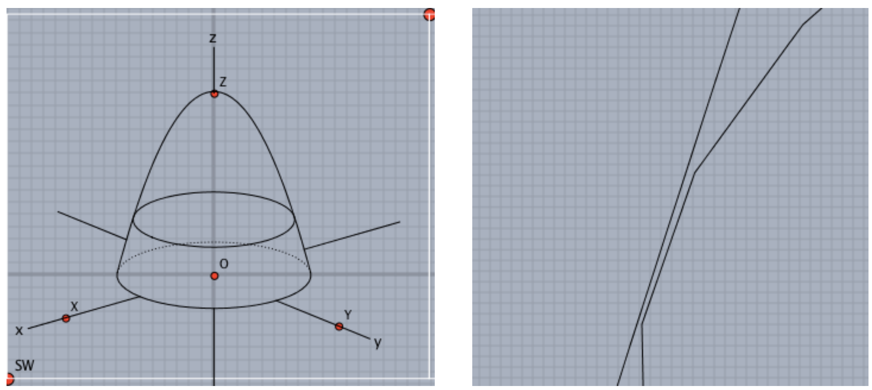

Fig.1

To refine the calculation of item 2, we have adopted an interpolatory scheme using Bézier curves near the contact point. Then we use a formula developed by Oshima[1] to decide the control points.

The left in the following is a further enlarged figure. The right shows Bézier curves in red color.

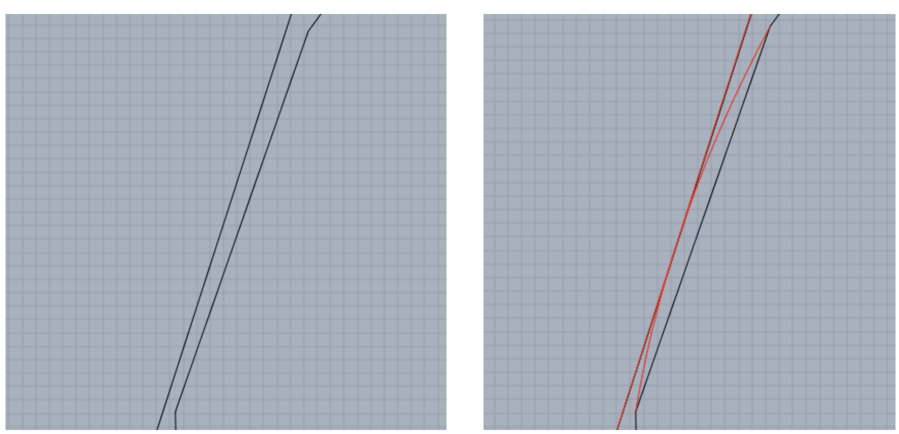

Fig.2

In this setting, the intersect is represented by a cluster of points. One of them is

$$\mathrm{P} = [-1.65827, 1.20578]. \tag{1}$$

KETCindy can also call Maxima from Cinderella and return a result back to Cinderella. For example, the intersect for Figure 2 is calculable using the following script in CindyScript. The result is:

$$\mathrm{P} = [-1.656701299244927, 1.210755779027779], \tag{2}$$

confirming that (1) is a very good approximation to the contact point.

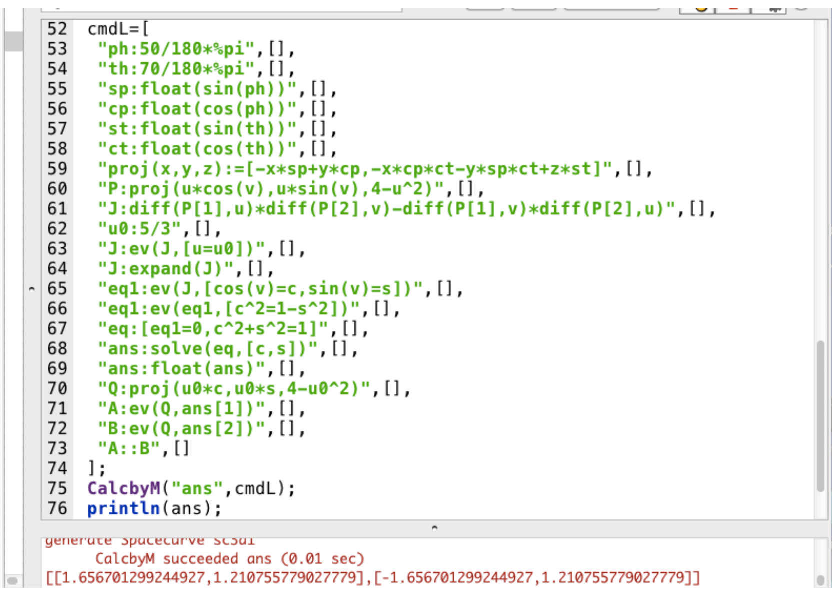

Fig.3

As a conclusion, we could say that the combined use of KETCindy, Cinderella, and Maxima is an effective tool to develop programs for surface drawing.

Acknowledgments

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