SPECIAL SESIONS

Applications of Computer Algebra - ACA2018

June 18–22, 2018

Santiago de Compostela, Spain

S10
Parametric Polynomial Systems

Thursday

Thu 21st, 16:30 - 17:00, Aula 10 – Antonio Montes:
Presentation of "The Grobner Cover"

Thu 21st, 17:30 - 18:00, Aula 10 – Yosuke Sato:
A canonical representation of continuity of the roots of a parametric zero dimensional multi-variate polynomial ideal

Thu 21st, 18:00 - 18:30, Aula 10 – Robert H. Lewis:
Fitting a Sphere to Point Cloud Data via Computer Algebra

Thu 21st, 18:30 - 19:00, Aula 10 – Robert H. Lewis:
Resultants, Implicit Parameterizations, and Intersections of Surfaces

Thu 21st, 19:00 - 19:30, Aula 10 – Cristina Bertone:
An overview on marked bases and applications

Friday

Fri 22nd, 10:30 - 11:00, Aula 10 – Katsusuke Nabeshima:
Computation methods of b-functions associated with μ-constant deformations –Case of inner modality 2–

Fri 22nd, 11:30 - 12:00, Aula 10 – Shinichi Tajima:
An effective method for computing Grothendieck point residues

Fri 22nd, 12:00 - 12:30, Aula 10 – Katsuyoshi Ohara:
An algorithm for computing Grothendieck local residues II –general case–

Organizers

Yosuke Sato:
Tokyo University of Science
Japan.

Katsusuke Nabeshima:
Tokushima University
Japan

Aim and cope

Parametric polynomial system solving is a challenge coming from many applications, such as biology, control theory, robotics, deformation of hypersurface singularities, etc. When a problem can be modelled by a parametric system, the main issue is not only to return its solutions, but also to describe them. The design of algorithms to solve parametric systems has recently become an active and expanding research field. Manipulating parametric systems is at the heart of computer algebra. It calls upon a wide range of methods, such as comprehensive Gröbner bases, Cylindrical Algebraic Decomposition, Quantifier Elimination, Comprehensive Triangular Sets, Comprehensive Involutive Systems, Parametric Local Cohomology System, etc.

This session is focused on the art of parametric system solving, for general class of systems or dedicated to specific application problems, including the following topics:

- Comprehensive Gröbner bases (systems)
- Quantifier elimination
- Comprehensive triangular sets
- Deformation of hypersurface singularities
- Modelisation of parametric problems
- Optimization of parametric systems
- Resolution of sparse parametric systems
- Low-level computation with multivariate polynomial coefficients
- Resolution of polynomial systems with boolean parameters
- Description of the real solutions of a parametric system
- Description of the parameter space of a polynomial system
- Extension of algorithms from non parametric to parametric systems

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An overview on marked bases and applications

Cristina Bertone[1]

The Hilbert scheme was introduced by Grothendieck in the 60s. One can simply think of the Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ as a set containing all the saturated homogeneous ideals I in a certain polynomial ring $\Bbbk[x_0, \dots, x_n] = \Bbbk[\mathbf{x}]$, with $\Bbbk$ a field, such that $\Bbbk[\mathbf{x}]/I$ has a given Hilbert polynomial $p(t)$. Grothendieck proved that $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ is not just a set, but it has a projective scheme structure. Although expert researchers investigated it, the Hilbert scheme is a mysterious object. Few properties are known, for instance Hartshorne proved connectedness in his Ph.D. Thesis.

A natural appealing application of Gröbner bases in Algebraic Geometry is the possibility to investigate families of ideals, and understand whether there is a scheme parameterizing them. In this framework, several authors tried to investigate the Hilbert Scheme by Gröbner techniques, see for instance [8]. The family of ideals having a certain initial ideal J for a given term order $\prec$ is called *Gröbner Stratum*. Imposing conditions for a suitable monic set of parametric polynomials to be a Gröbner basis gives the structure of closed scheme to the Gröbner Stratum of J in an affine space. Applying this construction to $J_{\geq r}$, where J is a monomial ideal such that $\Bbbk[\mathbf{x}]/J$ has Hilbert polynomial $p(t)$ and r is the Gotzmann number of $p(t)$, one can obtain a stratification of the Hilbert scheme by means of Gröbner Strata. Each of these Gröbner Strata is isomorphic to a locally closed subset (in general not an open subset) of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ [8, Theorem 6.3 (i)].

From the point of view of Algebraic Geometry, the fact that a Gröbner Stratum is not in general an open subset of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ is a big issue. This means that Gröbner Strata are not suitable to locally study $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$. Furthermore, it is not possible to obtain the ring of coordinates of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$, as a subscheme of a suitable projective space, by "glueing" the affine schemes of the Gröbner Strata that cover $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$.

In order to overcome the flaws of Gröbner strata with respect to the investigation of Hilbert schemes, a successful idea is to replace the use of a term order by considering special monomial ideals with strong combinatorial structure. Geometrically, it is totally reasonable to focus on this sort of monomial ideals: for instance Hartshorne proved the connectedness of the Hilbert scheme using *strongly stable* monomial ideals.

We construct families of ideals by suitable parametric polynomial generators which are monic in the terms generating a *quasi-stable ideal*. By imposing condition on these generators in order to have a *marked basis*, we describe an open subset of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ around the quasi-stable ideal.

1 Marked bases over a quasi-stable ideal

Here is a summary of the construction of marked bases over a quasi-stable ideal. The main references are [4, 7, 1].

Assume that $x_0 > \cdots > x_n$. If σ is a term, we denote by $\min(\sigma)$ (resp. $\max(\sigma)$) the index of the smallest (resp. biggest) variable dividing σ. We choose a *quasi-stable* monomial ideal $J \subset \Bbbk[\mathbf{x}]$. This monomial ideal has a special set of monomial generators, a *Pommaret basis* $\mathcal{P}(J)$, such that: for every $\sigma \in J$, there is a unique $\eta \in \mathcal{P}(J)$ such that $\sigma = \eta \cdot \delta$ where δ is a term and $\min(\eta) \geq \max(\delta)$.

Let A be a Noetherian $\Bbbk$-algebra. We construct a *set of (monic) marked polynomials over* J, $G_{\mathcal{P}(J)}$, in the following way: for every $\eta \in \mathcal{P}(J)$, we define $f_\eta := \eta - \sum_{\tau \notin J} c_{\eta\tau}\tau$, where $c_{\eta\tau} \in A$. The term η is the *head term* of f_η. The set $G_{\mathcal{P}(J)}$ is a *marked basis* if the terms of degree s outside J are a basis of the module $A[\mathbf{x}]_s/(G_{\mathcal{P}(J)})_s$, for every s. Thanks to the quasi-stability of J, it is possible to define a polynomial reduction process.

Definition 1.1. We denote by $\xrightarrow{G_{\mathcal{P}(J)}}$ the transitive closure of the following reduction relation in $A[\mathbf{x}]$: g and g' are in relation if $g' = g - c\delta f_\eta$, with $\delta\eta \in J$ is a term appearing in g with coefficient $c \neq 0_A$, f_η belongs to $G_{\mathcal{P}(J)}$, δ is a term and $\min(\eta) \geq \max(\delta)$.

The reduction $\xrightarrow{G_{\mathcal{P}(J)}}$ is Noetherian and confluent: for every $g \in A[\mathbf{x}]$, there is a unique h such that $g \xrightarrow{G_{\mathcal{P}(J)}} h$ and every term appearing with non-zero coefficient in h does not belong to J (the support of h is outside J).

Theorem 1.2. *[Buchberger-like criterion] For every $\eta \in \mathcal{P}(J)$, for every $i > \max(\eta)$, we compute $h_{\eta,i}$ such that $x_i f_\eta \xrightarrow{G_{\mathcal{P}(J)}} h_{\eta,i}$ and the support of $h_{\eta,i}$ is outside J. $G_{\mathcal{P}(J)}$ is a marked basis over J if and only if $h_{\eta,i} = 0$ for every $\eta \in \mathcal{P}(J)$, for every $i > \min(\eta)$.*

We can construct a marked set $\mathcal{G}_{\mathcal{P}(J)}$, replacing $c_{\eta\tau} \in A$ by a parameter $C_{\eta\tau}$. Let C be the set of paramters $C_{\eta\tau}$. By Theorem 6, we impose conditions in $\Bbbk[C]$ for $\mathcal{G}_{\mathcal{P}(J)}$ to be a marked basis: in this way we obtain a *marked scheme*. More precisely:

Theorem 1.3. *For every $\eta \in \mathcal{P}(J)$, for every $i > \min(\eta)$, compute $h_{\eta,i}$ as in Theorem 6. Let $\mathcal{R} \subset \Bbbk[C]$ be the ideal generated by the $\mathbf{x}$-coefficients of the polynomials $h_{\eta,i}$. The affine scheme $\mathrm{M}_{\mathcal{P}(J)} := \mathrm{Spec}(\Bbbk[C]/\mathcal{R})$ paramaterizes the ideals in $A[\mathbf{x}]$ generated by a marked basis over J, for every Noetherian $\Bbbk$-algebra A. We call* $\mathrm{M}_{\mathcal{P}(J)}$ marked scheme over J.

Marked schemes give an *open cover* of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ as follows. We compute the complete list L of saturated quasi-stable ideals J having Hilbert polynomial $p(t)$, and for each of them we compute the marked scheme over $J_{\geq r}$, where r is the *Gotzmann number* of $p(t)$. Each of these marked schemes is an open subset of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ [7,

Theorem 1.13].
Furthermore, we consider the usual action of $\mathrm{PGL} = \mathrm{PGL}_{\Bbbk}(\mathrm{n}+1)$ on $A[\mathbf{x}]$, and extend it to the points of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$. Up to this action of PGL, we get an open cover of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ by means of the computed marked schemes [7, Theorem 2.5]:

$$\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}} = \bigcup_{g \in \mathrm{PGL}, J \in L} g \cdot \mathrm{M}(J_{\geq r}). \tag{1}$$

This open cover is actually *functorial*: the marked schemes glue together, and it is possible to explicitely compute equations that define the projective scheme $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$ in a suitable projective space. This gives a new proof of the existence of the Hilbert scheme. The complete proof is in [6] for the Hilbert scheme, in [2] for the locus with bounded regularity, and in [1] this is generalized to Quot Schemes.

2 Some Applications

(1) The parametric system of equations we use to compute the conditions in $\Bbbk[C]$ for a marked basis is also used in order to study the liftings of a projective scheme. In [3], we prove that the liftings of a projective scheme with a given Hilbert polynomial are parameterized by a closed subscheme of a union of some marked schemes. Although Gröbner strata are sufficient to complete a first part of the investigation (x_n-liftings), marked schemes turn out to be the suitable approach to geometric liftings, due to the reasonable geometric assumption that the scheme to lift is in general position and to the openness of marked schemes in $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$.
(2) We can use marked schemes as open neighbourhoods of interesting points of $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(t)}}$, not only those defined by monomial ideals: for instance, we use them in [5] in order to prove the smoothability of the Gorenstein graded $\Bbbk$-algebras with Hilbert function $(1, 7, 7, 1)$ (and as a byproduct of the computations we obtain that Hilb^{7}_{16} has at least 3 irreducible components).
(3) As already mentioned, from the open cover (1), it is possible to compute the equations defining the Hilbert scheme as a subscheme of a suitable projective space [6]. This construction is generalized in [2] for the locus with bounded regularity, and in [1] to the case of Quot Schemes. These equations allow the direct study of Hilbert and Quot schemes. For instance, a paper on the Quot scheme of modules in $\Bbbk[x, y]^2$ with Hilbert polynomial $p(z) = 2$ is in progress.

Keywords: quasi-stable ideal, polynomial reduction process, Hilbert scheme

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Fitting a Sphere to Point Cloud Data via Computer Algebra

Robert H. Lewis[1], B. Paláncz[2] J. Awange[3]

To determine orientation using different kinds of sensors requires reference objects. One of the most frequently employed reference object is a sphere with known radius R and center coordinates (x, y, z).

In this paper we investigate the identification of these parameters from point cloud data contaminated by outliers and corrupted by low sensor resolution. Our main tools are Gröbner basis and the Dixon resultant. First the deterministic subsystems of the overdetermined system are solved. Algebraic computations show that when R is known, but the center coordinates are unknown, the algebraic and geometric fittings provide two solutions, while in the case of unknown R, the geometric fitting gives a unique solution.

The raw data of the point cloud were filtered using a Self Organized Map neural network. The overdetermined system was solved via a simplified Gauss-Jacobi technique using the results of the algebraic computations. This involves a polynomial system with 20 parameters. Our method is illustrated by a symbolic-numeric example based on real field measurement data using *Mathematica* and *Fermat* computer algebra systems.

Keywords: point cloud, polynomial system, resultant, symbolic-numeric, Gröbner basis

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Resultants, Implicit Parameterizations, and Intersections of Surfaces

Robert H. Lewis[1],

A classic problem in computer graphics and computer aided design is to derive an implicit equation for a surface given a parameterization of it. Since our surfaces are in three-dimensional space, we conventionally have three equations

$$\begin{aligned} x =& f(s,t) \\ y =& g(s,t) \\ z =& h(s,t) \end{aligned}$$

If homogeneous coordinates are being used, there is a fourth equation for w.

The implicit equation is produced by eliminating the s and t. As a very simple two-dimensional example, for a circle of radius r, the parametric equations are $x = r\cos(\theta), y = r\sin(\theta)$. It is easy to eliminate θ by squaring and adding:

$$x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$$

yielding the familiar equation for a circle. (r is not a variable, but a *parameter* in the other sense of the word “parameter.") Real examples of interest are much more complicated than this, and sophisticated elimination techniques are needed.

The simple example illustrates an important idea. Parametric systems frequently involve trig functions, usually sine and cosine. Elimination techniques usually require polynomial (or rational) functions. A system with sine and cosine is easily converted to a polynomial system by replacing cosine with, say, ct, sine with st, and adding a new equation $ct^2 + st^2 - 1 = 0$.

The theory of eliminating variables from a system of equations has a long history, starting with Bezout around 1760. A key idea is the *resultant* of a system of polynomial equations [2], [8]. Bezout did this for one-variable polynomials. Dixon in 1908 extended it to multivariate polynomials, and proved it would work in a certain ideal situation. However, for real problems the ideal situation rarely applies and often the method seems to fail. Kapur, Saxena, and Yang showed how to get around all those problems in 1994 [3]. Lewis refined and greatly improved the method in 2008 [4] to what is called Dixon-EDF. Gröbner bases can also be used to eliminate variables [8].

In spite of the 1994 publication, the Kapur-Saxena-Yang (KSY) method seems to have not been noticed by the computer graphics community. In 2000 the authors of [1] explicitly reject resultants as unworkable. In 2004 Wang [9] was aware of the Bezout-Dixon method but not KSY. He develops a new method to implicitize surfaces and tests fifteen examples with his method, resultants, and Gröbner bases. As in [1] he reports that in many cases resultants will not work because the Dixon method returns 0. This is one of the situations that KSY overcomes!

We compare Wang's reported time using pre-KSY Dixon, Wang's method, and our solution today using Dixon-KSY-EDF. We find our method to be greatly superior.

In 2017 Shen and Goldman [6] also report a new method for certain implicitizations. They also say that some resultant matrices have a 0 determinant and therefore resultants cannot be used. They do not refer to KSY.

We compare their reported times and our solutions today using Dixon-EDF working on some of their examples.

They try resultants in the generalized Sylvester form as found in [7] on their examples, and they also try Gröbner basis techniques. Gröbner bases failed in every case, meaning that nothing was returned within 10 minutes. Their resultants failed in the same way in every case except example 10.

Out techniques always work, are more efficient, and are more general.

In the following, Dixon always denotes the complete combination Dixon-KSY-EDF.

A second very important problem is to compute the intersection of two surfaces. Many papers have addressed this question. Virtually all the papers assume that the surfaces are *quadric*, i.e., degree 2. This means that the implicit equation is of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

We describe here an apparently new way to compute intersections so long as at least one of the surfaces is given by a conventional parameterization, as in the previous section. There is no restriction on the degrees of the surfaces, at least theoretically. Suppose surface one is given by

$$x = f_1(s_1, t_1), \;\; y = g_1(s_1, t_1), \;\; z = h_1(s_1, t_1)$$

and surface two is

$$x = f_2(s_2, t_2), \;\; y = g_2(s_2, t_2), \;\; z = h_2(s_2, t_2)$$

For the intersection simply combine this to form a system of six equations. Use Dixon to eliminate five variables, say y, z, t_1, s_2, t_2. That yields one equation (resultant) involving x and s_1. If this is linear in x, solve for x and obtain the parametric equation for the x-coordinate of the intersection curve. Repeat for y and z. One could just as well express x in terms of s_2, t_1 or t_2. That might have computational advantages.

The process described above also works if one surface has a parameterization and the second has an implicit definition, say $p(x, y, z) = 0$. We then have four equations $x = f_1(s_1, t_1), y = g_1(s_1, t_1), z = h_1(s_1, t_1), p(x, y, z) = 0$ and we eliminate three variables, say y, z, t_1.

If the resultant is degree 2 in x, one can easily use the quadratic formula to get two possible expressions for x in terms of s_1. Numerical testing could determine which is correct. Of course, degree 3 or 4 could also be handled by formulas, but the expressions would no doubt become daunting.

What if the degrees are higher than 2 or we don't want to deal with messy formulas? This leads to a new concept:

Definition: An implicit parameterization of a curve in 3-space is a set of three equations

$$f(x, s) = 0, \;\; g(y, s) = 0, \;\; h(z, s) = 0$$

whose solution set includes the curve. s is called the curve parameter.

Theorem Given two surfaces defined as above with polynomial functions, the Dixon resultant will produce an implicit parameterization of their intersection.

This follows immediately from the above discussion. The only possible flaw is if the set of six (or four) equations does not have a zero-dimensional solution space. That means for some values of the parameter s_1 there are infinitely many values of x. Dixon can fail in that case.

We will illustrate our techniques with many examples.

In summary,

- Computing an implicitization with Dixon is straightforward and routine. No special conditions on the surfaces are needed.
- The concept introduced here of "implicit parameterization" is easy to compute with Dixon. No special conditions on the surfaces are needed.
- Implicit parameterizations can be dealt with in fairly straightforward ways with commercial software.

Keywords: surface, polynomial system, resultant, Dixon, parameters, intersection, Gröbner basis

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Presentation of "The Gröbner Cover"

Antonio Montes[1]

I present the book "*The Gröbner Cover*" [6], that will be published during the present year. The contents are the following:

The genesis of this book is paper [7] for studying parametric polynomial systems.

Part 1 Theory: contains all the necessary tools to prove the existence and computation methods for obtaining the Canonical Gröbner Cover of a parametric polynomial system; Particularly, in Chapter 3, we provide the definitions and computation methods for obtaining all the canonical representations of constructible sets [3] and locally closed sets, that are used in Chapter 5 to obtain the Gröbner Cover, as well as for defining and computing all the algorithms provided in Part 2.

Part 2 Applications: contains three natural and interesting applications. Chapter 6 develops a new algorithm for Automatic Deduction of Geometric Theorems (`ADGT`) that, given a common geometric proposition of the form $(H \wedge \neg H_1) \Rightarrow (T \wedge \neg T_1)$, determines complementary hypothesis for the proposition to become a Theorem. The approach to this application was initiated in [5], but the new algorithm has not yet been published. Concerning Chapter 7, we introduced in [1] the taxonomy of the irreducible components of a Geometric Locus, which is determined by our `locus` algorithm. The content of Chapter 8, which has not yet been published either, generalizes the classical definitions, theorems and algorithms [2] for determining the envelope of a family of hyper-surfaces with more degrees of freedom than usual. Moreover, a new algorithm for determining the irreducible algebraic components of the envelope, as well as two other algorithms for approaching the real projection of the envelope are provided.

All the algorithms described in the text are implemented in the Singular library “grobcov.lib" [8], whose latest implementation can be downloaded from the web [4]. The book can also be used as a User Manual for the library.

In the talk I will present some examples using the new algorithms to show their utility and I will give a general outlook about the book.

Keywords: Parametric Polynomial System, Canonical Discussion, Parametric Gröbner System, Gröbner System.

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Computation methods of b-functions associated with μ-constant deformations – Case of inner modality 2 –

Katsusuke Nabeshima[1], Shinichi Tajima[2]

In this talk, computation methods of parametric b-functions are introduced for μ-constant deformation of quasihomogeneous singularities. The methods of b-functions associated with μ-constant deformations are constructed by using comprehensive Gröbner systems and the set of candidates of roots. In the cases of iner modality 2 ([7]), all b-funtions of associated with μ-constant deformations, can be obtained by our computation methods.

Let $\mathbb{C}\langle x, \partial_x\rangle$ denote the Weyl algebra, the ring of linear partial differential operators with coefficients in $\mathbb{C}$, where $x = (x_1, \dots, x_n)$, $\partial_x = (\partial_1, \dots, \partial_n)$, $\partial_i = \frac{\partial}{\partial x_i}$.

Let f be a non-constant polynomial in $\mathbb{C}[x]$. Then, the annihilating ideal of f^s is $\mathrm{Ann}(f^s) := \{p \in \mathbb{C}\langle s, x, \partial_x\rangle | pf^s = 0\}$ where s is an indeterminate. The *b-function* or the *Bernstein-Sato polynomial* of f is defined as the monic generator $b_f(s)$ of $(\mathrm{Ann}(f^s) + \mathrm{Id}(f)) \cap \mathbb{C}[s]$ where $\mathrm{Id}(f)$ is the ideal generated by f. It is known that the b-function of f always has $s+1$ as a factor and has a form $(s+1)\tilde{b}_f(s)$, where $\tilde{b}_f(s) \in \mathbb{C}[s]$. The polynomial $\tilde{b}_f(s)$ is called the *reduced b-function* of f.

It is known that a basis of the ideal $\mathrm{Ann}(f^s)$ can be computed by utilizing a Gröbner basis in $\mathbb{C}\langle x, \partial_x\rangle$ or PWB algebra ([5]). Moreorver, the reduced b-function $\tilde{b}_f(s)$ can be obtained by computing a Gröbner basis of $\mathrm{Ann}(f^s) + \mathrm{Id}(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

Let f be a parametric polynomial in $(\mathbb{C}[u])[x]$ where $u = (u_1, \dots, u_m)$ and u are parameters. In our previous paper [4], a computation method of comprehensive Gröbner systems (CGS) has been introduced in Poincare-Birkhoff-Witt (PBW) algebras. Thus, theoretically, a CGS of the ideal $\mathrm{Ann}(f^s)$ can be computed by utilizing the computation method. Moreover, a CGS of the ideal $\mathrm{Ann}(f^s) + \mathrm{Id}(f)$ can be computed, too. Hence, parametric b-functions can be computed by the following algorithm.

Algorithm 1.

Input: f: a parametric polynomial.
Output: reducded b-functions of f.
STEP 1: Compute a CGS of $\mathrm{Ann}(f^s)$.
STEP 2: Compute a CGS of $\mathrm{Ann}(f^s) + \mathrm{Id}(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

Algorithm 1 has been implemented in the computer algebra system Risa/Asir.

Table 1: reduced b-functions of $x^2z+yz^2+y^6+u_1y^4z+u_2z^3$

strata	reduced b-function
$\mathbb{C}^2\backslash\mathbb{V}(u_1)$	$B(s)(s+\frac{9}{8})(s+\frac{23}{24})$
$\mathbb{V}(u_1)\backslash\mathbb{V}(u_1,u_2)$	$B(s)(s+\frac{9}{8})(s+\frac{47}{24})$
$\mathbb{V}(u_1,u_2)$	$B(s)(s+\frac{17}{8})(s+\frac{47}{24})$

The Milnor number μ of the singularity $x^2z+yz^2+y^6=0$ is 17 (S_{17} singularity, the inner modality is 2), and the μ-constant deformation is given by $f=x^2z+yz^2+y^6+u_1y^4z+u_2z^3$ where u_1,u_2 are parameters. Our implementation can output Table 1 as the parametric reduced b-function of f within 5 hours where

$$B(s)=(s+\frac{3}{2})(s+\frac{4}{3})(s+\frac{7}{6})(s+\frac{11}{6})(s+\frac{7}{8})(s+\frac{11}{8})(s+\frac{13}{8})$$
$$\times(s+\frac{25}{24})(s+\frac{29}{24})(s+\frac{31}{24})(s+\frac{35}{24})(s+\frac{37}{24})(s+\frac{41}{24})(s+\frac{43}{24}).$$

Let us consider another example. The Milnor number of μ of the singularity $x^2z+yz^2+xy^4=0$ is 16 (S_{16} singularity, the inner modality is 2), and the μ-constant deformation is given by $f=x^2z+yz^2+xy^4+u_1y^6+u_2z^3$ where u_1,u_2 are parameters. In this case, our implementation of Algorithm 1 cannot return the parametric reduced b-function of f within "2 months". However, the implementation returns a CGS of $\mathrm{Ann}(f^s)$ within 1 day. Thus, we can infer that the computational complexity of $\mathrm{Ann}(f^s)+\mathrm{Id}(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})$ is quite big.

In order to avoid the big computation, Levandovskyy and Martin-Morales [3] have introduced a smart idea. We adopt the idea for computing b-functions of μ-constant deformations. However, the idea is not good enough to decide b-functions of μ-constant deformations. We need a further computation step that is checking local cohomology solutions of each holonomic D-module associted with a root of $\tilde{b}(s)=0$, to compute b-functions of μ-constant deformations.

In this talk, we introduce the further computation step and the new algorithm for computing b-functions associated with μ-constant deformations.

Let $f(u,x)=f_0+g\in(\mathbb{C}[u])[x]$ be a semi-quasihomogeneous polynomial, where f_0 is the quasihomogeneous part (or weighted homogeneous part) and g is a linear combination of upper monomials with parameters u. Then, f can be regard as a μ-constant deformation of f_0 with an isolated singularity at the origin. We have the foolowing classical results.

Theorem 1 Let $E_{f_0}=\{\gamma\in\mathbb{Q}|\tilde{b}_{f_0}(\gamma)=0\}$ where $\tilde{b}_{f_0}$ is the reduced b-function of f_0 on the origin. Then, for $e\in\mathbb{C}^m$, the set of roots of b-function of $f(e,x)$, on the origin, the set $E_{f(e,x)}=\{\gamma|b_{f(e,x)}(\gamma)=0\}$ becomes a subset of $E=\{\gamma-\ell\in\mathbb{Q}|\gamma\in E_{f_0},\ell\in\mathbb{Z},-n<\gamma-\ell<0\}$ where $\mathbb{Z}$ is the set of integers. That is, $E_{f(e,x)}\subset E$, for $e\in\mathbb{C}^m$.

Theorem 2 Let f be a non-constant polynomial in $\mathbb{C}[x]$, H a basis of $\mathrm{Ann}(f^s)$ in $\mathbb{C}\langle s,x,\partial_x\rangle$, $\gamma\in\mathbb{Q}$ and $r\in\mathbb{N}$. Let G be a minimal Gröbner basis of $\mathrm{Id}(H\cup$

$\{f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\} \cup \{(s-\gamma)^r\}\})$ w.r.t. a block term order $\succ$ s.t. $x \cup \partial_x \gg s$. Then, if $(s-\gamma)^r \in G$, $(s-\gamma)^r$ is a factor of the b-function of f.

The outline of the new algorithm is the following.

Algorithm 2.

Input: f: a parametric polynomial.
Output: reducded b-functions of f.
STEP 1: Compute a set E of candidates of roots of $\tilde{b}_f(s) = 0$.
STEP 2: Compute a CGS of Ann(f^s).
STEP 3: Compute a minimal Gröbner basis G of Ann(f^s) + Id$((s-\gamma)^r, f)$ (or Id$((s-\gamma)^r$,
$f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$)) in C$[s]\langle x, \partial_x\rangle$ where $\gamma \in E$ and $r \in$ N$_{>0}$.
If $(s-\gamma)^r \in G$, then $(s-\gamma)^r$ is a factor of the b-function of f.
STEP 4: For each stratum, check local cohomology solutions of each holonomic D-module associated with the root of $\tilde{b}_f(s) = 0$.

By executing Algorithm 2, we can obtain Table 2 as the parametric reduced b-function of $f = x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$ within 4 hours where

$$B(s) = (s+\tfrac{15}{17})(s+\tfrac{18}{17})(s+\tfrac{20}{17})(s+\tfrac{21}{17})(s+\tfrac{22}{17})(s+\tfrac{23}{17})(s+\tfrac{24}{17})(s+\tfrac{25}{17}) \\ \times(s+\tfrac{26}{17})(s+\tfrac{27}{17})(s+\tfrac{28}{17})(s+\tfrac{29}{17})(s+\tfrac{30}{17})(s+\tfrac{31}{17}).$$

Table 2: reduced b-functions of $x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$

strata	reduced b-function
$\mathbb{C}^2 \setminus \mathbb{V}(u_1)$	$B(s)(s+\frac{16}{17})(s+\frac{19}{17})$
$\mathbb{V}(u_1) \setminus \mathbb{V}(u_1, u_2)$	$B(s)(s+\frac{19}{17})(s+\frac{33}{17})$
$\mathbb{V}(u_1, u_2)$	$B(s)(s+\frac{33}{17})(s+\frac{36}{17})$

In this talk, we present mainly Algorithm 2 and show all b-functions of μ-constant deformation of inner modality 2.

Keywords: b-functions, comprehensive Gröbner systems, local cohomology

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An algorithm for computing Grothendieck local residues II — general case —

Katsuyoshi Ohara[1], **Shinichi Tajima**[2]

We will give an algorithm for exactly evaluating Grothendieck local residues for rational n-forms of n variables under general condition and show an implementation on a computer algebra system Risa/Asir. Grothendieck local residue is a natural generalization of the well-known residue for complex functions of single variable. The local residue was firstly described in Hartshorne [3] via the local duality in terms of derived category in much greater generality. The local duality can be also interpreted as a perfect pairing in terms of homological algebra. When a point is fixed, it can be realized as an integration of a meromorphic n-form of complex n variables on a real n-cycle around the point. Griffiths-Harris [2] described the following analytic definition of Grothendieck local residues.

Definition. Denote by $\mathcal{O}(U)$ a ring of holomorphic functions on a ball $U \subset \mathbf{C}^n$. Suppose that $f_1(x), \ldots, f_n(x) \in \mathcal{O}(U)$ make regular sequence and have only one isolated common zero $\beta \in U$. Let $\Gamma(\beta)$ be a real n-cycle around β defined by $\Gamma(\beta) = \{x \in U \mid \|f_1(x)\| = \varepsilon, \ldots, \|f_n(x)\| = \varepsilon\}$ and oriented by $d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0$. Denote $\tau_F = (f_1(x) \cdots f_n(x))^{-1} dx_1 \wedge \cdots \wedge dx_n$, where $x = (x_1, \ldots, x_n)$. For any $\varphi(x) \in \mathcal{O}(U)$, the integration

$$\mathrm{Res}_\beta(\varphi(x)\tau_F) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\Gamma(\beta)} \varphi(x)\tau_F$$

is called the *Grothendieck local residue* of meromorphic n-form $\varphi(x)\tau_F$.

The integration certainly gives an explicit representation of the local residue at the point β. However, in general, it is very hard to directly evaluate the integration because of complicated geometric shape of the real n-cycle in the $2n$-dimensional real space. To solve this problem, we use a method based on D-modules.

Let K be a subfield of $\mathbf{C}$ and denote $K[x] = K[x_1, \ldots, x_n]$. We suppose that a polynomial sequence $F = \{f_1, \ldots, f_n\}$ with K-coefficients is regular. The polynomial ideal I generated by F is zero-dimensional. The zero set $V_{\mathbf{C}}(I) = \{a \in \mathbf{C}^n \mid g(a) = 0,\ \forall g \in I\}$ is finite and it consists of isolated common zeros of the regular sequence F.

We introduce the n-th *algebraic local cohomology group* with support on $Z = V_{\mathbf{C}}(I)$ by

$$H^n_{[Z]}(K[x]) = \lim_{k\to\infty} \mathrm{Ext}^n_{K[x]}(K[x]/(\sqrt{I})^k, K[x]).$$

The algebraic local cohomology group $H^n_{[Z]}(K[x])$ can be regarded as a collection of equivalent classes of rational functions whose denominator has zero only on Z. Here the equivalence is given by cutting holomorphic parts of rational functions in a cohomlogical way.

According to the primary decomposition $I = \bigcap_{\lambda=1}^{\ell} I_\lambda$, the zero set also can be written as union of irreducible affine varieties: $Z = \bigcup_{\lambda=1}^{\ell} Z_\lambda$, where $Z_\lambda = V_{\mathbf{C}}(\sqrt{I_\lambda})$. Then $H^n_{[Z]}(K[x])$ is decomposed to direct sum

$$H^n_{[Z]}(K[x]) = H^n_{[Z_1]}(K[x]) \oplus \cdots \oplus H^n_{[Z_\lambda]}(K[x]) \oplus \cdots \oplus H^n_{[Z_\ell]}(K[x]).$$

Therefore an algebraic local cohomology class $\sigma_F = \left[\frac{1}{f_1 \cdots f_n}\right] \in H^n_{[Z]}(K[x])$ has unique decomposition

$$\sigma_F = \sigma_{F,1} + \cdots + \sigma_{F,\lambda} + \cdots + \sigma_{F,\ell},$$

where $\sigma_{F,\lambda} \in H^n_{[Z_\lambda]}(K[x])$. Note that $\mathrm{supp}\,(\sigma_{F,\lambda}) \subset Z_\lambda$. The decomposition above is a kind of partial fractional expansion of $\frac{1}{f_1 \cdots f_n}$ in terms of local cohomology.

Let $\beta \in Z_\lambda$ and $\varphi(x) \in \mathcal{O}(U)$ where U is a small neighborhood of β. We want to evaluate the local residue $\mathrm{Res}_\beta(\varphi\tau_F)$ where $\tau_F = (f_1(x) \cdots f_n(x))^{-1}dx$ and $dx = dx_1 \wedge \cdots \wedge dx_n$. If $j \neq \lambda$, then each $\sigma_{F,j}$ vanishes on U because $\mathrm{supp}\,(\sigma_{F,j}) \cap U = \emptyset$. Thus $\mathrm{Res}_\beta(\varphi\tau_F) = \mathrm{Res}_\beta(\varphi\, \sigma_{F,\lambda} dx)$ for $\beta \in Z_\lambda$. We denote by δ_{Z_λ} the local cohomology class which represents the delta function with the support Z_λ.

The algebraic local cohomology group can be naturally endowed with a structure of D-module. On the support Z_λ, from general theory, it follows $H^n_{[Z_\lambda]}(K[x]) = D_n \delta_{Z_\lambda}$. In other words, there exists a linear differential operator $T_{F,\lambda} \in D_n$ such that $\sigma_{F,\lambda} = T^*_{F,\lambda} \bullet \delta_{Z_\lambda}$ where $T^*_{F,\lambda}$ stands for the formal adjoint of $T_{F,\lambda}$. Since the local residue can be described in terms of local cohomology, we have $\mathrm{Res}_\beta(\varphi\tau_F) = \mathrm{Res}_\beta(\left[\frac{\varphi dx}{f_1 \cdots f_n}\right])$. Therefore

$$\begin{aligned}
\mathrm{Res}_\beta(\left[\frac{\varphi dx}{f_1 \cdots f_n}\right]) &= \mathrm{Res}_\beta(\varphi\sigma_F dx) \\
&= \mathrm{Res}_\beta(\varphi \cdot (T^*_{F,\lambda} \bullet \delta_{Z_\lambda})dx) \\
&= \mathrm{Res}_\beta((T_{F,\lambda} \bullet \varphi) \cdot \delta_{Z_\lambda} dx) \\
&= (T_{F,\lambda} \bullet \varphi)|_{x=\beta}.
\end{aligned}$$

That is, the mapping $\varphi \mapsto \mathrm{Res}_\beta(\varphi\tau_F)$ is determined by the differential operator $T_{F,\lambda}$. Since the set $\{(T_{F,\lambda}, Z_\lambda) \mid \lambda = 1, 2, \ldots, \ell\}$ gives the Grothendieck local residue mapping, the local residue of any meromorphic n-forms can be evaluated by differential operators $T_{F,\lambda}$. Our purpose is to find the differential operator $T_{F,\lambda}$ without the use of an explicit representative element of the local cohomology class $\sigma_{F,\lambda}$.

Under certain condition for the regular sequence F, we already gave an algorithm for computing differential operators $T_{F,\lambda}$ (see [6]). We have extended the method for more general setting. In this talk, we will describe new algorithm and show an implementation on the computer algebra system. Our algorithm consists of the following steps.

1. Find the primary decomposition $I = \bigcap_{\lambda=1}^{\ell} I_\lambda$.
2. Find the annihilating left-ideal $\mathrm{Ann}_{D_n}(\sigma_F)$.
3. For each λ, find the vector space V_λ over $K[x]/\sqrt{I_\lambda}$ spanned by Noether differential operators of the associated prime $\sqrt{I_\lambda}$.
4. For each λ, find a "monic" operator $S_\lambda^* \in V_\lambda$ such that $\mathrm{Ann}_{D_n}(\sigma_F)S_\lambda^* \subset \mathrm{Ann}_{D_n}(\delta_{F,\lambda})$.
5. For each λ, determine the differential operator $T_{F,\lambda}^*$ from S_λ^*.

Keywords: Local residues, Local Cohomology, Holonomic System

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A canonical representation of continuity of the roots of a parametric zero dimensional multi-variate polynomial ideal

Yosuke Sato[1], **Ryoya Fukasaku**[2], **Hiroshi Sekigawa**[3]

In [2, 3], we introduced the following result **Theorem 1** which gives a sufficient condition of a generator of a multivariate parametric zero dimensional ideal for the continuity property of its roots. In [3], using the result we also give a correctness proof of an algorithm for real quantifier elimination one of the authors has recently developed and implemented in [1]. In this talk, using the theory introduced in [4], we show the following results **Theorem 2** and **Theorem 3** which enable us both to describe and to compute a canonical representation form of continuity of the roots of a given parametric zero dimensional multi-variate polynomial ideal.

In what follows, $\bar{A} = A_1, \ldots, A_m$ and $\bar{X} = X_1, \ldots, X_n$ denote variables, we consider $\bar{A}$ as parameters $\bar{X}$ as main variables. The symbol $\succ$ denotes an admissible term order on the set of all terms of $\bar{X}$, for a polynomial f in $\mathbb{Q}[\bar{A}, \bar{X}]$, $LM(f)$, $LT(f)$ and $LC(f)$ denote the leading monomial, the leading term and the leading coefficient of f respectively regarding f as a member of the polynomial ring over the coefficient ring $\mathbb{Q}[\bar{A}]$, i.e. $f \in (\mathbb{Q}[\bar{A}])[X]$.

Definition 1. *Let S be an algebraically constructible subset of an affine space $\mathbb{C}^m$ for some natural number m. A finite set $\{\mathcal{S}_1, \ldots, \mathcal{S}_k\}$ of non-empty subsets of S is called an algebraic partition of S if it satisfies the following properties 1, 2 and 3:*

1. $\cup_{i=1}^{k} \mathcal{S}_i = S$.
2. $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ *if* $i \neq j$.
3. $\mathcal{S}_i$ *is a locally closed set for each i, that is $\mathcal{S}_i = V_{\mathbb{C}}(I_1) \backslash V_{\mathbb{C}}(I_2)$ for the varieties $V_{\mathbb{C}}(I_1), V_{\mathbb{C}}(I_2)$ of some ideals I_1, I_2 of $\mathbb{Q}[\bar{A}]$.*

Each $\mathcal{S}_i$ is called a segment.

Definition 2. *Let S be an algebraically constructible subset of $\mathbb{C}^m$. For a finite subset F of $\mathbb{Q}[\bar{A}, \bar{X}]$, a finite set $\mathcal{G} = \{(\mathcal{S}_1, G_1), \ldots, (\mathcal{S}_k, G_k)\}$ satisfying the following properties 1, 2, 3 and 4 is called a comprehensive Gröbner system of F over S with parameters $\bar{A}$ w.r.t. $\succ$:*

1. *Each G_i is a finite subset of* $\mathbb{Q}[\bar{A}, \bar{X}]$.
2. $\{\mathcal{S}_1, \ldots, \mathcal{S}_k\}$ *is an algebraic partition of S.*

3. *For each $\bar{c} \in \mathcal{S}_i$, $G_i(\bar{c}) = \{g(\bar{c}, \bar{X}) | g(\bar{A}, \bar{X}) \in G_i\}$ is a Gröbner basis of the ideal $\langle F(\bar{c})\rangle$ in $\mathbb{C}[\bar{X}]$ w.r.t. $\succ$, where $F(\bar{c}) = \{f(\bar{c}, \bar{X}) | f(\bar{A}, \bar{X}) \in F\}$.*

4. *For each $\bar{c} \in \mathcal{S}_i$, $LC(g)(\bar{c}) \neq 0$ for any element g of G_i.*

In addition, if each $G_i(\bar{c})$ is a minimal (reduced) Gröbner basis, $\mathcal{G}$ is said to be minimal (reduced). Being monic is not required. When $\mathcal{S}$ is the whole space $\mathbb{C}^m$, the words "over $\mathcal{S}$" is usually omitted.

The following fact is one of the most important properties of a minimal comprehensive Gröbner system.

Fact 1. $LT(G_i(\bar{c}, \bar{X}))$ is identical for each $\bar{c} \in \mathcal{S}_i$. Hence, the dimension of a $\mathbb{C}$-vector space $\mathbb{C}[\bar{X}]/\langle G_i(\bar{c}, \bar{X})\rangle$ is invariant for $\bar{c} \in S_i$ if it is finite. Consequently, when the $\mathbb{C}$-vector space has dimension l for each $\bar{c} \in S_i$, the ideal $\langle G_i(\bar{c}, \bar{X})\rangle$ has l number of roots in $\mathbb{C}^n$ counting their multiplicities.

Considering the above roots as a l size multiset and introducing a natural topology on a set of the same size multisets, we have the following property.

Theorem 1. Let $\mathcal{G} = \{(\mathcal{S}_1, G_1), \ldots, (\mathcal{S}_k, G_k)\} \subset \mathbb{Q}[\bar{X}, \bar{A}]$ be a minimal comprehensive Gröbner system with parameters $\bar{A}$ w.r.t. an arbitrary term order of main variables $\bar{X}$. If the ideal $\langle G_i(\bar{c})\rangle$ is zero dimensional for each $\bar{c} \in \mathcal{S}_i$, then the set of all roots of the system of the parametric polynomial equations $g(\bar{A}, \bar{X}) = 0, g \in G_i$ is continuous in the segment $\mathcal{S}_i$ as a function of the parameters $\bar{A}$.

Note that the multisets of the roots of two ideals $\langle G_i(\bar{a})\rangle$ and $\langle G_j(\bar{b})\rangle$ for $\bar{a} \in \mathcal{S}_i$ and for $\bar{b} \in \mathcal{S}_j$ may have the same size for some different i, j, even when $LT(G_i(\bar{c}, \bar{X}))$ and $LT(G_j(\bar{c}, \bar{X}))$ are distinct. For such a case we still have the following property.

Theorem 2. Using the same notations in the previous theorem, if the multisets of the roots of two ideals $\langle G_i(\bar{c})\rangle$ and $\langle G_j(\bar{c})\rangle$ have the same size but $LT(G_i(\bar{c}, \bar{X}))$ and $LT(G_j(\bar{c}, \bar{X}))$ are distinct, then two segments $\mathcal{S}_i$ and $\mathcal{S}_j$ are not path-connected.

This property enables us to describe a canonical representation form of continuity of the roots of a given parametric multi-variate polynomial ideal as follows.

Theorem 3. Given a finite set F of $\mathbb{Q}[\bar{A}, \bar{X}]$ and a term order $\succ$. There exists a unique partition $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$ of $\mathbb{C}^m$ such that the following properties hold.

1. Each $\mathcal{A}_i$ is an algebraically constructible set.

2. $LT(\langle F(\bar{c}, \bar{X})\rangle)$ is invariant for $\bar{c}$ on each $\mathcal{A}_i$.

3. $LT(\langle F(\bar{a}, \bar{X})\rangle)$ and $LT(\langle F(\bar{b}, \bar{X})\rangle)$ are distinct if $\bar{a} \in \mathcal{A}_i$ and $\bar{b} \in \mathcal{A}_j$ for different i, j.

4. If $\langle F(\bar{c}, \bar{X})\rangle$ has finite zeros in $\mathbb{C}^n$ for $\bar{c}$ on $\mathcal{A}_i$, the map from $\mathcal{A}_i$ to the set of multisets of such zeros is continuous.

5. If $\mathbb{C}[\bar{X}]/\langle F(\bar{a},\bar{X})\rangle$ and $\mathbb{C}[\bar{X}]/\langle F(\bar{b},\bar{X})\rangle$ have the same finite dimension as $\mathbb{C}$ vector space for $\bar{a} \in \mathcal{A}_i$ and $\bar{b} \in \mathcal{A}_j$ $(i \neq j)$, then $\mathcal{A}_i$ and $\mathcal{A}_j$ are not path-connected.

Remark 1. Using the theory of [4], we can also compute such a partition $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$ from a given finite set F of $\mathbb{Q}[\bar{A},\bar{X}]$.

Remark 2. The partition $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$ depends on the choice of a term order, however, it seems to be independent, though we have not shown it yet.

Keywords: Comprehensive Gröbner System, Representation of Continuity, Quantifier Elimination

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An effective method for computing Grothendieck point residues

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In this talk, we present an effective algorithm for computing Grothendieck local residues associated to semi-quasi homogeneous hypersurface isolated singularities. The key idea of our approach is the use of Grothendieck local duality.

The theory of Grothendieck local residue is a cornerstone of algebraic geometry and complex analysis. It has been used in diverse problems of several different fields ofmathematics. It is known theoretically that the classical transformation law given in [2] can be used to compute its values. Whereas computing Grothendieck local residue is quite difficult even if one uses computer algebra systems, because of the cost of computation in local rings. Developing effective methods for computing has been desired in many applications.

We consider in this talk Grothendieck point residues associated to a μ-constant deformation of quasi-homogeneous hypersurface isolated singularity. Based on the theory of local cohomology and Grothendieck local duality, we propose a new effective method for computing Grothendieck local residues. A key innovation of the resulting algorithm is an improvement of a previous algorithm on extended ideal membership problems in the ring of convergent power series [5].

To be more precise, let $f(x,t) = f_0(x) + g(x,t)$ be a semi-quasi homogeneous polynomial, where $f_0(x)$ is the quasi-homogeneous part, $g(x,t) = \sum_{j=1}^{\ell} t_j x^{\beta_j}$ is a sum of upper monomials with $x = (x_1, x_2, ..., x_n)$ main variables, $t = (t_1, t_2, ..., t_\ell)$ deformation parameters. Set $F = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}]$ and let $\tau_F = \left[\begin{array}{c} 1 \\ \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \end{array} \right]$ denote the local cohomology class in $\mathcal{H}^n_{\{\mathcal{O}\}}(\mathcal{O}_X)$ with parameters t where $\left[\begin{array}{c} 1 \\ \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \end{array} \right]$ is the Grothendieck symbol.

Let $res_{\{\mathcal{O}\}}(h, \tau_F dx)$ denote the Grothendieck point residue at the origin $\mathcal{O}$ in $\mathbf{C}^n$ of the differential form

$$\frac{h(x)}{\frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

where $h(x)$ is a germ of holomorphic function. The linear map which assigns to each holomorphic function $h(x)$ the Grothendieck point residue

$$h(x) \longrightarrow res_{\{\mathcal{O}\}}(h, \tau_F dx)$$

can be expressed in terms of partial differential operators. Namely there exists a linear partial differential operator T, s.t.

$$(Th)(\mathcal{O}) = res_{\{\mathcal{O}\}}(h, \tau_F dx)$$

By using algorithm for computing algebraic local cohomology classes with parameters ([4]) , we introduce an effective method for computing the linear partial differential operator T. We also show the resulting algorithm for computing Grothendieck point residues associated to a μ-constant deformation of quasi homogeneous hypersurface isolated singularity.

We present some examples of computation.

Keywords: Grothendieck local residue, local cohomology, Grothendieck local duality

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