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Parametric integrals, combinatorial identities and applications

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We propose a survey of parametric integrals (aka sequences of definite integrals) studied by undergraduates in an engineering school and pre-service teachers, in a technology-rich environment. The papers in reference provide a few examples only. Parametric integrals are interesting both for their mathematical properties, and the numerous applicable methods, and for their importance in applied science; see [1].

Let be given an either definite or improper integral of the so-called second type $I_n = \int_a^b f_n(t)\, dt$, where $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ are given. The study of the family of integrals I_n can yield the following results:

1. An induction formula for the sequence (I_n), such as $I_{n+1} = R(n)I_n$ or $I_{n+1} = u_n + R(n)I_n$, etc., where $R(n)$ is a function of the parameter n; see [2,3,4,5,6].
2. A closed formula for I_n as a function of the parameter n, often using telescoping methods which lead to factorial expressions. This is the case if $R(n)$ is a rational function. In some cases, induction connects I_{n+2} and I_n, and the usage of double factorials may yield more compact formulas. Otherwise, the study of the convergence of a series is necessary.
3. Combinatorial identities, in the case where more than one integration method can be applied.
4. New integral presentations of classical combinatorial numbers; see [3,4,5]

Technology contributes to the study in various ways.

1. A Computer Algebra System provides often an interactive tutor for integration methods. Its usage for small values of the parameter helps to find a general way to compute I_n as a function of $I_{n-1}, I_{n-2}, \ldots$. With this, closed formulas can be looked for.

2. The Online Encyclopedia of Integer Sequences (`oeis.org`). Experiments with the CAS provide the first terms of the sequences of integrals. Using the database, candidates to describe the sequence (I_n) are obtained. Determination of a closed formula is made easier.

Specific situations may appear:

- The computation of the integral for general parameter may be performed directly by the CAS. This has been the case for $I_n = \int_0^{\pi/2} \frac{dt}{1+tan^n(t)}$ with DERIVE (it returns immediately $\pi/4$, an answer independent of the value of the parameter!). Other CAS had hard time with this integral. The reason is that a specific theorem is implemented there; this theorem does not appear in most textbooks and is explained in [1].
- If the answer is readable immediately, we are done. The answer may involve special functions. For example, if $I_n = \int_0^{\pi/2} \sin^n t \; dt$, Maple's command returns immediately $I_n = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\frac{n}{2})}{2\,\Gamma(1+\frac{n}{2})}$, providing an incitement to learn something new, the Gamma function, as an extension of the curriculum. An example is described in [5].

We illustrate the different cases with new examples of integrals of rational functions, trigonometric functions, etc., and examples of applications in science and engineering.

Keywords
Parametric integrals, combinatorics, applications

References
[1] G. Boros and V. Moll, *Irresistible Integrals: Symbolics, Analysis And Experiments In The Evaluation Of Integrals*, Cambridge University Press (2004).
[2] Th. Dana-Picard, Parametric integrals and symmetries of functions, *Mathematics and Computer Education*, 5-12 (2005).
[3] Th. Dana-Picard, Integral presentations of Catalan numbers and Wallis formula, *International Journal of Mathematical Education in Science and Technology* **42** (1), 122-128 (2011).
[4] Th. Dana-Picard and D.G. Zeitoun, Sequences of definite integrals, infinite series and Stirling numbers, *International Journal of Mathematical Education in Science and Technology* **43** (2), 219-230 (2012).
[5] Th. Dana-Picard and D.G. Zeitoun, Parametric integrals, Wallis formula and Catalan numbers, *International Journal of Mathematical Education in Science and Technology* **43** (4), 515-520 (2012).
[6] Th. Dana-Picard and D.G. Zeitoun, Exploration of Parametric Integrals related to a Question of Soil Mechanics, *International Journal of Mathematical Education in Science and Technology* **48** (4), 617-630 (2017).
[7] P. Glaister, Factorial sums, *International Journal of Mathematical Education in Science and Technology* **34** (2), 250-257 (2003).