Parametric integrals, combinatorial identities and applications

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We propose a survey of parametric integrals (aka sequences of definite integrals) studied by undergraduates in an engineering school and pre-service teachers, in a technology-rich environment. The papers in reference provide a few examples only. Parametric integrals are interesting both for their mathematical properties, and the numerous applicable methods, and for their importance in applied science; see [1].

Let be given an either definite or improper integral of the so-called second type

\[ I_n = \int_a^b f_n(t) \, dt, \]

where \( a, b \in \mathbb{R} \) and \( n \in \mathbb{N} \) are given. The study of the family of integrals \( I_n \) can yield the following results:

1. An induction formula for the sequence \((I_n)\), such as \( I_{n+1} = R(n)I_n \) or \( I_{n+1} = u_n + R(n)I_n \), etc., where \( R(n) \) is a function of the parameter \( n \); see [2,3,4,5,6].

2. A closed formula for \( I_n \) as a function of the parameter \( n \), often using telescoping methods which lead to factorial expressions. This is the case if \( R(n) \) is a rational function. In some cases, induction connects \( I_{n+2} \) and \( I_n \), and the usage of double factorials may yield more compact formulas. Otherwise, the study of the convergence of a series is necessary.

3. Combinatorial identities, in the case where more than one integration method can be applied.

4. New integral presentations of classical combinatorial numbers; see [3,4,5]

Technology contributes to the study in various ways.

1. A Computer Algebra System provides often an interactive tutor for integration methods. Its usage for small values of the parameter helps to find a general way to compute \( I_n \) as a function of \( I_{n-1}, I_{n-2}, \ldots \). With this, closed formulas can be looked for.
2. The Online Encyclopedia of Integer Sequences (oeis.org). Experiments with the CAS provide the first terms of the sequences of integrals. Using the database, candidates to describe the sequence \((I_n)\) are obtained. Determination of a closed formula is made easier.

Specific situations may appear:

- The computation of the integral for general parameter may be performed directly by the CAS. This has been the case for \(I_n = \int_0^{\pi/2} \frac{dt}{1 + \tan^n(t)}\) with DERIVE (it returns immediately \(\pi/4\), an answer independent of the value of the parameter!). Other CAS had hard time with this integral. The reason is that a specific theorem is implemented there; this theorem does not appear in most textbooks and is explained in [1].

- If the answer is readable immediately, we are done. The answer may involve special functions. For example, if \(I_n = \int_0^{\pi/2} \sin^n t \, dt\), Maple’s command returns immediately \(I_n = \frac{\sqrt{\pi} \Gamma\left(\frac{n+2}{2}\right)}{2 \Gamma\left(\frac{n+1}{2}\right)}\), providing an incitement to learn something new, the Gamma function, as an extension of the curriculum. An example is described in [5].

We illustrate the different cases with new examples of integrals of rational functions, trigonometric functions, etc., and examples of applications in science and engineering.

Keywords
Parametric integrals, combinatorics, applications

References