Familiarizing students with definition of Lebesgue integral using Mathematica - some examples of calculation directly from its definition: Part 2

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In popular books of calculus, for example [2, 3], we can find many examples of Riemann integral calculated directly from its definition. The aim of these examples is to familiarize students with the definition of Riemann integral. In this article, with similar aim but for Lebesgue integral definition, we present the following examples of calculation directly from its definition:

\[ \int_0^1 x \chi_Q(x) \, dm(x), \int_0^\infty e^{-x} \, dm(x), \int_0^1 (-\ln x) \, dm(x), \int_1^\infty \frac{1}{x} \, dm(x), \int_0^1 \frac{1}{x} \, dm(x) \]

and some others, \( dm(x) \) denotes the Lebesgue measure on the real line. The title of this talk is very similar to the title of author’s article [1] in which there are examples of Lebesgue integrals of bounded function over bounded intervals calculated directly from its definition but in our talk we show examples of Lebesgue integrals of bounded or unbounded function over bounded or unbounded intervals calculated directly from its definition. We calculate sums, limits and plot graphs of needed simple functions using Mathematica. Using Mathematica or others CAS programs for calculation Lebesgue integral directly from its definitions, seems to be didactically useful for students because of the possibility of symbolic calculation of sums, limits - checking our hand calculations and plotting dynamic graphs. Moreover we get students used not only to the definition of Lebesgue integral but also to CAS applications generally.

The two following definitions of Lebesgue integral are used in this article:

Let \((\mathbb{R}, \mathcal{M}, m)\) be measure space, where \(\mathcal{M}\) is \(\sigma\)-algebra of Lebesgue measurable subsets in \(\mathbb{R}\), and \(m\)-Lebesgue measure on \(\mathbb{R}\).

Let \(f : \mathbb{R} \to \mathbb{R}\) be measurable nonnegative function (we’ve omitted the definition of Lebesgue integral for simple real measurable functions).

**Definition 1.** (See [4, 6, 7, 8, 9])

\[ \int f \, dm(x) = \sup \left\{ \int s \, dm(x) : 0 \leq s \leq f, s \text{ simple measurable function} \right\}. \quad (1) \]
Definition 2. (See [5, 10, 11]) Let $s_n$ be nondecreasing sequence of nonnegative simple measurable functions such that $\lim_{n \to \infty} s_n(x) = f(x)$ for every $x \in \mathbb{R}$. Then:

$$\int f \, dm(x) = \lim_{n \to \infty} \int s_n \, dm(x).$$

(2)

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Higher education, Lebesgue integral, Application of CAS, Mathematica, Mathematical didactics

References