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Automated exploration of envelopes and offsets with networking of technologies

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Envelopes of parameterized families of plane curves, of space curves, of surfaces, are an important topic both because of the mathematics involved and because of their applications (e.g. the determination of safety zones around sprinklers, robotic plants, Luna Park attractions, etc.). A drawback of this domain is the small number of its theorems, and the need to study numerous special cases [10]. Moreover, there exists 4 non-equivalent definitions of envelopes; see [3, 1].

The usage of technology makes the study of envelopes a live domain of study and may attract students to exploration and discovery (e.g., see [4, 5, 6]). A Dynamic Geometry System (DGS) provides an environment for automated exploration and discovery. In particular GeoGebra's companion package GeoGebra Discovery has a command for the determination of an envelope under certain conditions for the construction [7]. Nevertheless, the commands may not work in certain situations (such as non-polynomial data or higher degree polynomials). It may be then useful to transfer the data (the equations) to a Computer Algebra System, with which analytic solutions will be computed. The output may be afterwards transferred back to the DGS.

Let a parameterized family of plane curves $\mathcal{C}_t$ be given by the equation $F(x, y, t) = 0$. If an envelope exists, it is given by the solution of the system of equations

$$\begin{cases} F(x, y, t) = 0, \\ \frac{\partial F}{\partial t} F(x, y, t) = 0. \end{cases}$$

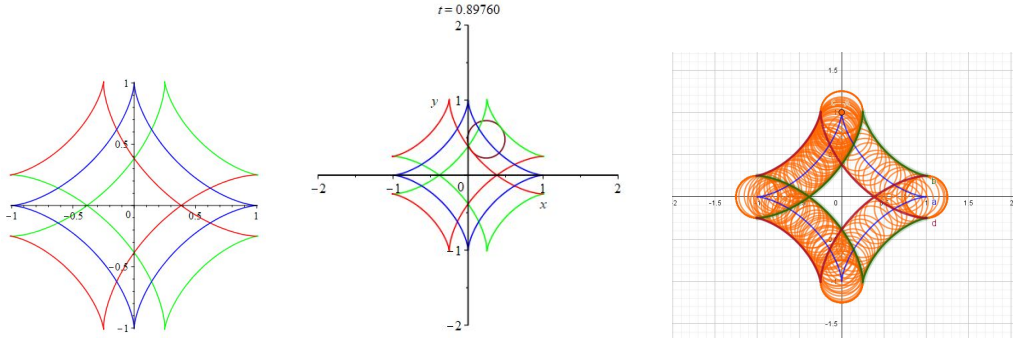
In a polynomial setting, the **solve** command of the CAS uses algorithms from the theory of Gröbner bases [8]. In various situations, it is possible to transform the given data into polynomial form. Then the CAS provides a parametric presentation of the envelope (which can be described as the disjoint union of several components). These equations are copied into the DGS for the final graphical presentation (e.g. using the **Curve** command of GeoGebra).

In this talk:

1. We show how this “networking” of technologies is used;

2. We analyze the differences between the animations provided by the CAS and the interactive exploration enabled by the DGS, and how to have profit of these differences.

3. We analyze the possible contradiction between the first intuition and the actual output, in particular with regards to the issue of safety zones evoked above. In Figure 1 we show a family of circles centred on an astroid. The envelope of the family is different from the hull enclosing all the circles in the family.



(a) (b) (c)

Figure 1: Envelope vs safety hull

Figure 1(a) shows the astroid and the envelope of a family of circles centred on it, with radius 1/4, after computations with Maple and implicit plot. Figure 1(b) is a snapshot of an animation obtained with Maple, Figure 1(c) shows the output of a mouse driven experimentation with GeoGebra, using in GeoGebra algebraic results from Maple, networking with the technologies. Figures 1(a)and 1(c) reveal different aspects: the last one has been obtained with an interactive exploration using the DGS. The algebraic description has been obtained as an offset of the astroid, namely the geometric locus of the points constructed as follows: for each point on the astroid, consider the normal at this point and the point at distance $\frac{1}{4}$ out of the astroid. The important difference between the 1^{st} and the last Figures is the arcs of circles appearing around the cusps the astroid. The analysis of these arcs requires strong zooming.

In [6], we studied offsets of a deltoid, here we perform similar work based on an astroid. Once again, new constructions of interesting plane curves appear.

The features and activities that we describe here show how to implement and develop the 4 C’s of Education in the 21^{st} century [9]: Critical thinking, Creativity, Communication and Collaboration. If the first two C’s are human characteristics, the exploration that we propose requires the two last C’s both for humans and for machines and expands also the man-and-machine C’s. Strong zooming is a must in order to have an accurate conjecture of what happens, in particular regarding singular points.

Keywords
Automated exploration, Envelopes, Networking, 4 C’s of Education

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