Applications of Computer Algebra – ACA 2021
Virtual. Online | July 23-27, 2021
Session on *"Session Title"*

A CAS-DGS assisted exploration of Spiric curves and their Hessians

Thierry Dana-Picard[1] [ndp@jct.ac.il]

[1] Department of Mathematics, Jerusalem College of Technology, Jerusalem, Israel

During the last decades, several packages have been developed, both in Computer Algebra Systems (CAS) and in Dynamic Geometry Systems (DGS) for the study of plane algebraic curves. Using them, the study became experimental, involving graphics and animations, and algebraic computations. It involves exploration, discovery and then proof, all this based on a fruitful dialog between the kinds of software [4]. Even when the student does not master all the theoretical material, software may provide a bypass of the problem [1] and also incite to learn more mathematics.

We study here the points of inflexion of special biquartic curves called *spiric curves*, which appear as (b)isoptic curves of conics [2,3] . They can be realized as the intersection of a torus with a plane parallel to the torus's axis (recall that a torus is generated by revolving a circle around an axis). *Cassini ovals* are a special case of these spirics; they have been conjectured by the astronomer Cassini as a model for planetary motion. Kepler's law finally took the central place, and people claimed that these ovals lost their importance and became a pure geometric object. Actually, they still appear in various scientific domains, such as electrostatics, and isoptic curves. Generally in the literature, the revolving circle does not intersect the axis and a general form of the equation of the curve is

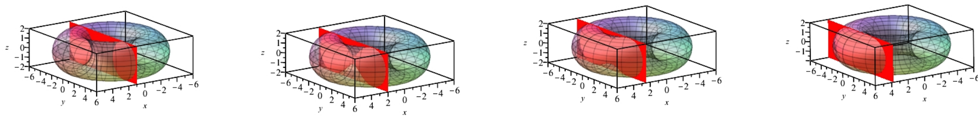
$$(x^2+y^2)^2 - 2a(x^2-y^2) + a^2 - b^2 = 0, \tag{1}$$

where a and b are positive real parameters.The limiting case where $a = 0$ is a (double) circle. In the situation described by Equation (1), the torus is a regular one, i.e. non self-intersecting:

1. If $a > b$, the curve is the union of two loops;
2. For $a = b$, the curve is a lemniscate;
3. If $a < b$, the curve is a single loop, which may have points of inflexions or not.

These cases reflect the distance from the center of the revolving circle to the axis; see Fig. 1.

In [4], offsets of these curves are explored and new constructions are shown. The needed dialog between a Computer Algebra System (CAS) and a Dynamic Geometry System (DGS) is analyzed there, in order to explore, conjecture and prove new results.

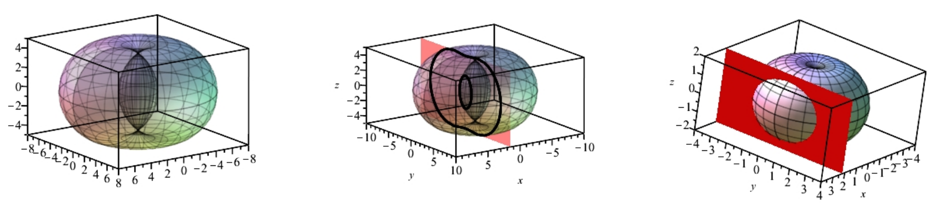


(a) (b) (c) (d)

Figure 1: Toric intersections

We work in a more general case, where the torus is self-intersecting, i.e. the revolving circle intersects the axis. These are the curves appearing as bisoptic curves of conics. In [1], the pair torus-plane is reconstructed from the data of the curve.

Depending on the distance from the center of the revolving circle to the rotation axis, the intersection may have either only one component, or two disjoint components. The distance center-axis influences also the existence of points of inflexion on the spiric. See Figure 1.



(a) (b) (c)

Figure 2: Plane intersection with a self-intersecting torus

The existence of points of inflexion is explored using the following theorem: for a given plane curve $\mathcal{C}$ given by an equation of the form $F(x, y) = 0$, an associate curve is defined by the vanishing of the so-called *Hessian determinant*:

$$\begin{vmatrix} \frac{\partial^2}{\partial x^2}F(x,y) & \frac{\partial^2}{\partial x \partial y}F(x,y) \\ \frac{\partial^2}{\partial y \partial x}F(x,y) & \frac{\partial^2}{\partial x^2}F(x,y) \end{vmatrix}$$

We will call this curve the Hessian of $\mathcal{C}$.

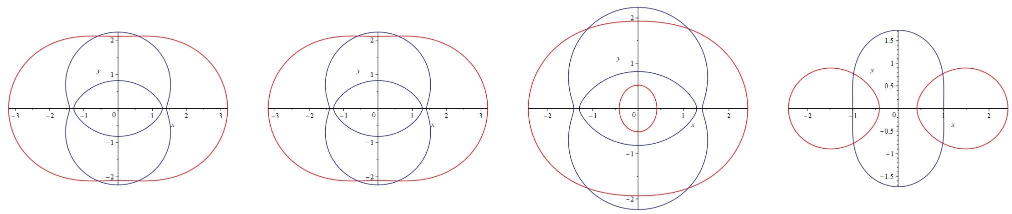
Using the command **Hessian** of Maple 2021, we show easily that the Hessian of a spiric curve is also a spiric curve. A couple of examples of the pair curve-Hessian is displayed in Figure 3, using the general equation

$$(x^2 + y^2)^2 + ax^2 + by^2 + c = 0, \tag{2}$$

where $a, b, c \in \mathbb{R}$. The equation of the Hessian is then

$$(x^2 + y^2)^2 + + \left(\frac{1}{6}a + \frac{1}{2}b\right) x^2 + \left(\frac{1}{6}b + \frac{1}{2}a\right) y^2 + \frac{1}{12}ab \tag{3}$$

Figure 3 show a few examples. Other configurations exist. In each case, the triple (a, b, c) is given; the original spiric is in red and its Hessian in blue.



(a) $(a,b,c) = (-8,-4,-2)$ (b) $(a,b,c) = (-10,-4,-2)$ (c) $(a,b,c) = (-6,-4,1)$ (d) $(a,b,c) = (-6,4,1)$

Figure 3: A spiric and its Hessian

Copying the data into a DGS* a hand-driven exploration is performed to check the points of inflexion, but a precise determination of these points requires the algebraic abilities of the CAS. A well-known theorem states that if $\mathcal{C}$ has points of inflexion, they are points of intersection of $\mathcal{C}$ with its Hessian; see [5,6]. According to the background of the students, various commands can be used, starting from **solve** to solve almost manually the system of equations (the output is often given using teh plac holder *RootOf* and the command **allvalues** has to be applied), and including **intersectcurves** (from the **algcurves** package). Checking whether the points of intersection which are determined are points of inflexion or not requires knowledge on curves given by an implicit equation which is not always taught. This work emphasizes once again the importance of the dialog between the educator and the technologies, and between the technologies, studied in [4].

We explore the relation between the shape of the original oval $\mathcal{C}$ and of its Hessian $\mathcal{H}$, also the relation between the two pairs of intersecting torus-plane,using the method of [1].

Keywords
Automated exploration, spiric curves, Cassini ovals, Hessian, inflexion points

References
[1] TH. DANA-PICARD. Technology as a bypass for a lack of theoretical knowledge, International Journal of Technology in Mathematics Education 11 (3), 101–109 (2005).
[2] TH. DANA-PICARD, G. MANN AND N. ZEHAVI. *From conic intersections to toric intersections: the case of the isoptic curves of an ellipse*, The Montana Mathematical Enthusiast 9 (1), 59–76 (2011).
[3] TH. DANA-PICARD, N. ZEHAVI AND G. MANN. *Bisoptic curves of hyperbolas*, International Journal of Mathematical Education in Science and Technology 45 (5), 762–781 (2014).
[4] TH. DANA-PICARD AND Z. KOVÁCS. *Offsets of Cassini ovals*, Preprint, 2021.
[5] J.J. CALLAHAN. *Advanced Calculus: A Geometric View*. Springer Science & Business Media, 2010.
[6]A.A. SWAMINATHAN. *Inflection Points in Families of Algebraic Curves*, Thesis, Harvard College, 2017. Downloadable: `https://www.math.harvard.edu/media/swaminathan.pdf`

*We used GeoGebra, freely downloadable from `http:\geogebra.org`