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Is computer algebra ready for conjecturing and proving geometric inequalities in the classroom?

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Supporting automated reasoning in the classroom has a long history in the era of computer algebra. Several systems have been developed and introduced as prototypes at various school levels during the last decades. A breakthrough in using computers to obtain automated proofs is still expected, even if some freely available systems offer easy access to such technical means.

In teaching geometry we refer to *GeoGebra* which became the de facto standard of a handy geometry toolset in many schools worldwide. It allows conjecturing and proving *equational* statements, including the geometric properties like parallelism, perpendicularity or equality of lengths of segments in a planar geometric figure [9, 13], and more recently, *inequational* theorems [16].

It is well-known that proving geometric inequalities is a more difficult scenario. Even if there were robust frameworks created in the last 30 years including *QEPCAD B* [6, 3], Reduce/*Redlog* [8], Maple/*RegularChains* [4], Maple/*SyNRAC* [10] or *Mathematica* [18], practical use of them was not yet in the focus of educational research. In our talk we introduce an extension of GeoGebra by adding a layer that is capable of using QEPCAD B (via the *Tarski* [17] system) to conjecture and prove, or directly prove some simple geometric inequalities.

In our extension we build on the classical way of translating the geometric setup into an algebraic system, based on the revolutionary work of Wu [19], Chou [5], and improved later by Recio and Vélez [14] with elimination theory. On the other hand, we partly use general purpose real quantifier elimination (RQE) methods to find the best possible geometric constants to conjecture and prove sharp inequalities between two expressions. Our implementation uses cylindrical algebraic decomposition (CAD) that promotes effective RQE. In fact, the translation of the geometric setup sometimes involves inequalities well, for example, when a point is put on a segment or inside a triangle, or angle bisectors are drawn.

Fig. 1 shows how the inequality $s \leq 3\sqrt{3}R$ (where s stands for the semiperimeter and R for the circumradius) can already be mechanically proven by our toolset in an intuitive way. The

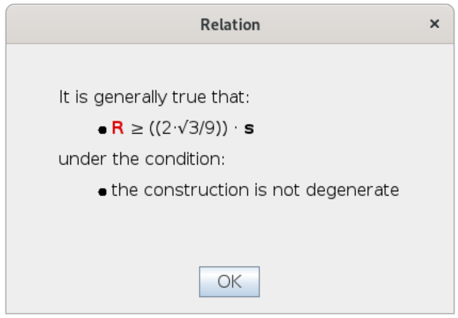

Figure 1: A simple inequality that is automatically discovered and proven by GeoGebra Discovery

underlying semi-algebraic translation is shown if Fig. 2.

In our communication we do not go into the hidden technical difficulties, but refer to the paper [16] that focuses on the RQE related issues, and points to a large set of benchmarks based on our tool (including several examples from [1]). Instead, we focus on the practical use: how our work can be fruitful for the student and the teacher in a classroom.

Our experimental system is already capable of solving a large set of open questions in planar Euclidean geometry. But speed remains an important issue: we recall that solving a CAD problem has doubly exponential complexity in the number of variables (see [2, 7]).

GeoGebra Discovery is freely available at [11] for all three majors platforms (for Linux a 64 bit version and a Raspbian variant are published).

In our communication we will reflect on the potential impact of GeoGebra Discovery in the educational world. We will mention self-experimenting as well as student and teacher trainings to prepare for mathematics contests and exams. To illustrate our concept we will show some examples that are based on the books [1] and [15].

Keywords
Automated deduction in geometry, Inequalities, GeoGebra.

References

[1] O. Bottema, R. Djordjevic, R. Janic, D. Mitrinovic, and P. Vasic. *Geometric Inequalities*. Wolters-Noordhoff Publishing, Groningen, 1969.

[2] C. Brown and J. Davenport. The complexity of quantifier elimination and cylindrical algebraic decomposition. In *Proceedings of ISSAC '07*, pages 54–60. ACM, 2007.

[3] C. W. Brown. An overview of QEPCAD B: a tool for real quantifier elimination and formula simplification. *Journal of Japan Society for Symbolic and Algebraic Computation*, 10(1):13–22, 2003.

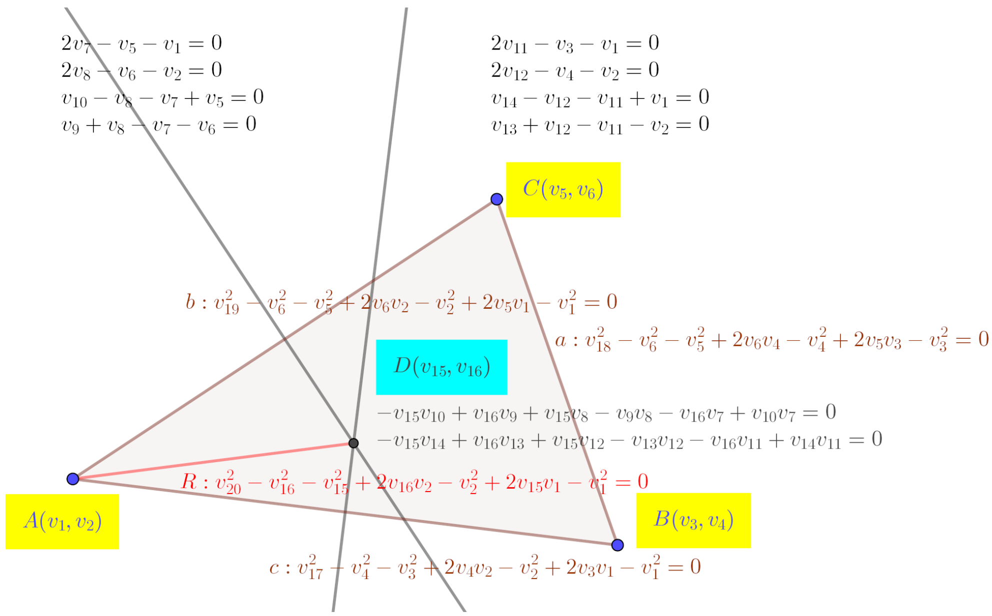

Figure 2: Translation of the hypotheses of the geometric problem setting into an algebraic (or semi-algebraic) system

[4] C. Chen and M. M. Maza. Quantifier elimination by cylindrical algebraic decomposition based on regular chains. *Journal of Symbolic Computation*, 75:74–93, 2016.

[5] S. C. Chou. *Mechanical Geometry Theorem Proving*. D. Reidel Publishing Company, Dordrecht, Netherlands, 1988.

[6] G. E. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12:299–328, 1991.

[7] J. Davenport and J. Heintz. Real quantifier elimination is doubly exponential. *Journal of Symbolic Computation*, 5(1).

[8] A. Dolzmann and T. Sturm. Redlog: Computer algebra meets computer logic. *ACM SIGSAM Bulletin*, 31(2):2–9, 1997.

[9] M. Hohenwarter, Z. Kovács, and T. Recio. *Using Automated Reasoning Tools to Explore Geometric Statements and Conjectures*, pages 215–236. Springer International Publishing, Cham, 2019.

[10] H. Iwane, H. Yanami, and H. Anai. SyNRAC: a toolbox for solving real algebraic constraints. In *Proceedings of ICMS-2014. LNCS, vol. 8592*.

[11] Z. Kovács. GeoGebra Discovery. A GitHub project, 07 2020. `https://github.com/kovzol/geogebra-discovery`.

[12] Z. Kovács, T. Recio, P. R. Richard, S. V. Vaerenbergh, and M. P. Vélez. Towards an ecosystem for computer-supported geometric reasoning. *International Journal of Mathematical Education in Science and Technology*, 2020.

[13] Z. Kovács, T. Recio, and M. P. Vélez. Using automated reasoning tools in GeoGebra in the teaching and learning of proving in geometry. *International Journal of Technology in Mathematics Education*, 25(2):33–50, 2018.

[14] T. Recio and M. P. Vélez. Automatic discovery of theorems in elementary geometry. *Journal of Automated Reasoning*, 23:63–82, 1999.

[15] I. Reiman. *Fejezetek az elemi geometriából*. Typotex, Budapest, 2002.

[16] R. Vajda and Z. Kovács. GeoGebra and the *realgeom* reasoning tool. *CEUR Workshop Proceedings*, pages 204–219, 6 2020.

[17] F. Vale-Enriquez and C. Brown. Polynomial constraints and unsat cores in TARSKI. In *Mathematical Software – ICMS 2018. LNCS, vol. 10931*, pages 466–474. Springer, Cham, 2018.

[18] Wolfram Research, Inc. Mathematica, version 12.1, 2020. Champaign, IL.

[19] W. T. Wu. On the decision problem and the mechanization of theorem proving in elementary geometry. *Scientia Sinica*, 21:157–179, 1978.