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Can I bring my calculator to the exam? Some reflections on the abstraction level of CAS

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For many years, the standard answer to the question "Can I bring my calculator to the exam?" (regarding the college entrance exams in Spain –denoted EvAU [1]) was: "Yes. But it will be useless".

From the maths teachers' point of view, we could classify the questions of maths exams in four classes:

- **Numerical computations and substitutions in formulae:**
  They arise in all fields of maths (algebra, geometry, analysis, astronomy,...). They were solved in the past with pen and pencil and sometimes using logarithm tables, slide rules, etc. They can be solved with a (classic) calculator.
  Example 1: Calculate the final price of a skirt tagged a price of 23.5 Euros (VAT excluded) if the VAT is 21% and it has a 10% discount.

- **Symbolic computations (simplifications, expansions, concatenation of algebraic calculations, etc.):**
  Example 2: Simplify $(x + y)^2 - (x - y)^2$.
  Example 3: Analyse when a certain parameter-depending linear system has solutions and find them in such case.
  Example 4: Draw a function by computing its zeroes, maxima and minima, asymptotes, inflection points, etc. In the EvAU these functions are normally trickily chosen so that they have, for instance, two very close zeroes [2] (in order a graphic calculator to be useless).
  Example 5: Find the integral $\int x \cdot log(x)\, dx$ (it is straightforward using the integration by parts method).
  In the past they could only be solved by hand. Now these tasks can be completed by computer algebra systems (CAS), that are available for computers, tablets and smartphones.

- **Theoretical questions regarding mathematical formulae:**
  Example 6: What is the cube of a binomial.
  Example 7: What is $sin(2x)$ equal to?
  Example 8: What is the determinant of a 3×3 matrix (Sarrus rule)?
  Example 9: What is the derivative of the product of two functions?
  All the mentioned tasks can also be completed by a CAS.

- **Theorem proving:**
  There are two main lines regarding research in theorem proving: logic deduction from the axioms (using deduction rules), applicable to all fields of mathematics, and automatic theorem proving in geometry [3] (using algebraic methods) [4–6]. CAS are the key tool for the latter line of research. The main problem is the readability of the proofs produced by the two lines of research aforementioned and their lack of elegance (synthetic proofs and proofs based on brilliant ideas can't be developed these ways). But this is not the topic of this talk.

From the CAS point of view (unlike what happens from the teacher's point of view) there is no difference between the questions in the second and third classes: they can be completed by the CAS just performing symbolic computations. They are of an abstraction level [7] higher than those of the first class (as they deal with non-assigned variables –variables in the mathematical sense, not in the computational sense).

The case of Example 9 is specially interesting, as it reaches an even higher abstraction level: the CAS deals in this case with general functions, not only with already declared functions, themselves depending on non-assigned variables. Many maths teachers and CAS users are unaware of such possibility of CAS. It can be easily introduced to, for instance, *Maple*[*] [8–12] and computed by this CAS:

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> diff(f(x)*g(x),x);
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$$\left(\frac{d}{dx}\, f(x)\right) g(x) + f(x) \left(\frac{d}{dx}\, g(x)\right)$$

Summarising, CAS have reached an unprecedented abstraction level with many possibilities in different fields. As a consequence, assessment in maths education depends on the availability of technological tools: a CAS can be used to solve symbolic problems and also as a technological live cheat sheet (in theoretical questions).

Keywords
Computer algebra, Abstraction level, Assessment.

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[*]*Maple* is a trademark of Waterloo Maple Inc.

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