Lectures

Trace ideals in commutative algebra and combinatorics

Jürgen Herzog

University of Duisburg-Essen, Germany

Lecture 1: On the set of trace ideals of a Noetherian ring

We introduce trace ideals of modules and describe their basic properties. We classify the one-dimensional analytically irreducible local Gorenstein rings which admit only a finite number of trace ideals. This is joint work with Masoomeh Rahimbeigi.

Lecture 2: Nearly Gorenstein rings

Nearly Gorenstein rings were introduced by Takayuki Hibi, Dumitru Stamate and myself. They are the local Cohen–Macaulay rings with the property that the trace of the canonical module contains the maximal ideal. The canonical trace for tensor products and Segre products of Gorenstein algebras, as well as of (squarefree) Veronese subalgebras will be considered. The results are used to classify the nearly Gorenstein Hibi rings. Also the relationship between nearly Gorenstein and almost Gorenstein will be discussed. This is a report on joint work with Hibi and Stamate.

Lecture 3: Measuring the non-Gorenstein locus of Hibi rings and normal affine semigroup rings

In this lecture I report on joint work with Janet Page and Fatemeh Mohammadi in which we describe the non-Gorenstein locus of Hibi rings and of normal affine semigroup rings. The trace of the canonical module of a Segre product of algebras which are not necessarily Gorenstein will be considered, and the results will be applied to compute the non-Gorenstein locus of toric rings. Several necessary and sufficient conditions will be given for Hibi rings and normal semigroup rings to be Gorenstein on the punctured spectrum.

Lecture 4: The colength of the canonical trace and far-flung Gorenstein rings

For a numerical semigroup ring H, the colength of the canonical trace is compared with $g(H)-n(H)$, where $g(H)$ is the number of gaps of H and $n(H)$ is the number of non-gaps of H. One-dimensional Cohen-Macaulay rings whose canonical trace is as small as possible are those whose colength is as large as possible. Such rings are called far-flung Gorenstein rings. Far-flung Gorenstein rings will be considered especially for numerical semigroup rings. It is shown that the solution of the

Rohrbach problem in additive number theory provides an upper bound for the multiplicity of far-flung Gorenstein numerical semigroup rings. This lecture covers joint work with Dumitru Stamate and Shinya Kumashiro.

Lecture 5: Rings of Teter type

A 0-dimensional local ring $(R, \mathfrak{m}_R)$ is called a Teter ring, if there exists a local Gorenstein ring $(G, \mathfrak{m}_G)$ such that $R \cong G/(0 : \mathfrak{m}_G)$. This class of rings has been introduced 1974 by William Teter. It has been shown by Huneke and others that a local ring R is a Teter ring if and only if there exists an epimorphism $\varphi : \omega_R \to \mathfrak{m}_R$. Here ω_R denotes the canonical module of R. The ring is called of Teter type, if there exists a surjective homomorphism from ω onto the trace of ω_R. Various classes of algebras of Teter type will be considered. This is joint work with Oleksandra Gasanova, Takayuki Hibi and Somayeh Moradi.

Convex polytopes, Gröbner bases and monomial ideals

Takayuki Hibi

Osaka University, Japan

In the current trends of commutative algebra, the role of combinatorics is distinguished and indispensable. Half a century ago, combinatorics of the modern theory of convex polytopes had created a fascinating research area in commutative algebra.
In the series of my talks, first, the historical background of how combinatorics was introduced into the community of commutative algebra in the late 1970s will be reviewed, and second, an overview of the influence of the Gröbner basis theory in commutative algebra via toric rings and toric ideals in the early 1990s will be surveyed. Finally, the prominent topics of monomial ideals and toric rings in the quarter century since the late 1990s will be discussed. A tentative schedule of the series of my talks is as follows:

- Face enumeration of convex polytopes
- Cohen–Macaulay complexes and triangulations
- A quick introduction to Gröbner bases
- Hibi rings and Hibi ideals
- Castelnuovo polytopes

No special knowledge on combinatorics and Gröbner bases will be required to understand my talks. However, it is desirable for the audience to be familiar with Cohen–Macaulay rings together with linear resolution.

Binomial ideals associated to graphs

Sara Saeedi Madani

Amirkabir University of Technology (Tehran Polytechnic) and IPM, Iran

In these lectures, we give an introduction to some binomial ideals associated to graphs, like binomial edge ideals and Hankel edge ideals of graphs. We give an overview on several algebraic and homological properties and invariants of those binomial ideals which have been mainly investigated in terms of the underlying graphs in some interesting ways.

Contributed Talks

Sanghoon Baek

Korea Advanced Institute of Science and Technology (KAIST), South Korea

Chow ring of generic flag varieties under the spin groups

For a smooth variety X, the canonical epimorphism from the Chow ring of X to the associated graded ring of the topological filtration on the Grothendieck ring of X is not in general injective. When X is a generic flag variety under a semisimple group G, Karpenko conjectured that the morphism is an isomorphism. In this talk, we will discuss some counter-examples to this conjecture for spin groups G.

Carmelo Cisto

University of Messina, Italy

Gröbner basis of closed path polyominoes

A polyomino $\mathcal{P}$ is a connected collection of cells of $\mathbb{Z}^2$. We denote by $V(\mathcal{P})$ the set of all vertices of the cells belonging to $\mathcal{P}$ and considering $a, b \in \mathbb{Z}^2$, with $a < b$ with respect to natural partial order, the set $[a, b] = \{x \in \mathbb{Z}^2 \mid a \leq x \leq b\}$ is called an *inner interval of* $\mathcal{P}$ if it is contained in $\mathcal{P}$. In [6], A. A. Qureshi showed how to associate a coordinate ring to a collection of cells $\mathcal{P}$. In particular, set $S = K[x_v | v \in V(\mathcal{P})]$, where K is a field, and if $[a, b]$ is an inner interval of $\mathcal{P}$, with a,b and c,d respectively diagonal and anti-diagonal corners, the binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $\mathcal{P}$. Let $\mathcal{M}$ be the set of all inner 2-minors of $\mathcal{P}$, then the *polyomino ideal* of $\mathcal{P}$ is the ideal $I_{\mathcal{P}}$ generated by $\mathcal{M}$, and $K[\mathcal{P}] = S/I_{\mathcal{P}}$ is the *polyomino ring* of $\mathcal{P}$. An active area of research is to study the main algebraic properties of $K[\mathcal{P}]$ depending on the geometric configuration of $\mathcal{P}$.

Among the most interesting properties to investigate there is the structure of the Gröbner basis of $\mathcal{P}$ with respect to different monomial orders on S. For instance, in [6] the author shows under which conditions $\mathcal{M}$ is the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to the monomial order $<^1_{\text{lex}}$ on S, defined as the lexicographic order induced by the total order on $V(\mathcal{P})$ given by $a <^1 b$ if and only if, for $a = (i, j)$ and $b = (k, l)$, $i < k$, or $i = k$ and $j < l$. Other classes of monomial orders having a similar fashion are defined and studied in [3].
We are interested in studying the Gröbner basis of $\mathcal{P}$ for a particular class of polyominoes that we call *closed paths*. We show that, depending on the structure of the closed path $\mathcal{P}$, it is always possible to obtain that the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to a monomial order$<^Y_{\text{lex}}$, defined for a suitable $Y \subset V(P)$, is the set $\mathcal{M}$. As a consequence, we obtain that in the case $\mathcal{P}$ is a closed path polyomino such that $I_{\mathcal{P}}$ is a prime ideal (a characterization for the primality of closed paths is provided in [1]) then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain.

References

[1] C. Cisto, F. Navarra, Primality of closed path polyominoes, Journal of Algebra and its Applications, in press, 2021.

[2] C. Cisto, F. Navarra, R. Utano, On Gröbner bases and Cohen-Macaulay property of closed path polyominoes, arXiv:2205.08251, submitted.

[3] C. Mascia, G. Rinaldo, F. Romeo, Primality of polyomino ideals by quadratic Gröbner basis, Matematische Nachrichten, 295:593–606, 2022

[4] A.A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, J. Algebra 357, 279–303, 2012.

Antonino Ficarra

University of Messina, Italy

Polymatroidal ideals and their homological shifts

Matroid theory is one of the most fascinating research areas in Combinatorics. In 2003, based on the polyhedral theory on integral polymatroids developed in the sixties, Herzog and Hibi introduced the concept of discrete polymatroids as a multiset analogue of matroids. Since then, polymatroidal ideals have been a main topic in Combinatorial Commutative Algebra and are the source of many ispiring conjectures. One of the most outstanding such conjectures is due to Bandari, Bayati and Herzog. It predicts that all the homological shift ideals of a polymatroidal ideal are again polymatroidal. In the present talk, the recent progresses on this conjecture will be surveyed. A partial recent positive result obtained by the author will be presented. Finally, the general theory of homological shift ideals will be discussed as well as some open questions in this reaserch topic.

Francesca Gandini

University of Primorska, Slovenia

Equivariant Hilbert series of subspace arrangements

To an hyperplane arrangement we can associate a combinatorial object, a matroid. Similarly, to a subspace arrangement we associate a polymatroid. Each subspace in the arrangement can be viewed algebraically as a linear ideal. We can study the product of these linear ideals by using the combinatorial data of the polymatroid. In particular, by tensoring the ambient vector space with an n-dimensional vector space, we introduce an action of GL_n on the subspace arrangement and consequently on the product of linear ideals. The equivariant Hilbert series of the product of the linear ideals can be recursively constructed from the polymatroid of the subspace arrangement. Moreover, considering the transpose functor on the category of polynomial representations, we produce an ideal in the exterior algebra associated to the subspace arrangement. The corresponding effect of this functor on the equivariant Hilbert series is the involution that maps the Schur symmetric function s_λ to $s_{\lambda'}$, the function indexed by the transpose of the original partition.

Aslı Musapaşaoğlu

Sabancı University, Turkey

Dominating ideals and closed neighbourhood ideals of graphs

We study the closed neighborhood ideals and the dominating ideals of graphs, in particular, some classes of trees and cycles. We prove that the closed neighborhood ideals and the dominating ideals of some classes of trees are normally torsion free. The closed neighborhood ideals and the dominating ideals of cycles fail to be normally torsion free. However, we prove that the closed neighborhood ideals of cycles admit the (strong) persistence property and the dominating ideals of cycles are nearly normally torsion free. This is joint work with Ayesha Asloob Qureshi, Mehrdad Nasernejad and Somayeh Bandari.

Francesco Navarra

University of Messina, Italy

Hilbert-Poincaré series and rook polynomial of closed path polyominoes

A *polyomino* is a finite collection of unitary squares joined edge by edge. In 2012 A. A. Qureshi established a connection between polyominoes and Commutative Algebra, attaching to a polyomino $\mathcal{P}$ the ideal generated by all inner 2-minors of $\mathcal{P}$ in a suitable polynomial ring S (see [6]). This ideal is called a *polyomino ideal* of $\mathcal{P}$ and it is denoted by $I_\mathcal{P}$. Let $K[\mathcal{P}] = S/I_\mathcal{P}$ be the *coordinate ring* of $\mathcal{P}$. In the study of the Poincaré-Hilbert series of $K[\mathcal{P}]$, an interesting conjecture has recently emerged, which states that a polyomino $\mathcal{P}$ is thin, which means that $\mathcal{P}$ does not contain a square tetromino, if and only if the h-polynomial of $K[\mathcal{P}]$ is equal to the rook polynomial of $\mathcal{P}$ (see [4]).

The rook polynomial of $\mathcal{P}$ is a polynomial $r_{\mathcal{P}}(t) = \sum_{i\in\mathbb{N}} r_i t^i$ whose coefficient r_i represents the number of distinct ways to arrange i rooks on the cells of $\mathcal{P}$ in non attacking positions. However, it seems quite difficult to give a complete proof of this conjecture, as soon as to study the Poincaré-Hilbert series of $K[\mathcal{P}]$ for a generic polyomino $\mathcal{P}$ (see also [3],[5],[7]).
Inspired by these considerations, in [2] we study the reduced Poincaré-Hilbert series of the coordinate ring attached to a closed path, a kind of non-simple thin polyomino introduced in [1]. In particular, we prove that the h-polynomial of $K[\mathcal{P}]$, where $\mathcal{P}$ is a prime closed path, is the rook polynomial of $\mathcal{P}$, obtaining as a consequence the regularity and the Krull dimension of $K[\mathcal{P}]$. Finally we characterize the Gorenstein prime closed paths by the S-property, showing that the coordinate ring attached to a closed path $\mathcal{P}$ without zig-zag walks is Gorenstein if and only if $\mathcal{P}$ consists of maximal blocks of length three.

References

[1] C. Cisto, F. Navarra, Primality of closed path polyominoes, Journal of Algebra and its Applications, in press, 2021.

[2] C. Cisto, F. Navarra, R. Utano, Hilbert-Poincaré series and Gorenstein property for closed path polyomioes, arXiv:2205.08375, submitted.

[3] V. Ene, J. Herzog, A. A. Qureshi, F. Romeo, Regularity and Gorenstein property of the L-convex Polyominoes, The Electronic Journal of Combinatorics, **28**(1), P1.50, 2021.

[4] G. Rinaldo, and F. Romeo, Hilbert Series of simple thin polyominoes, J. Algebr. Comb. **54**, 607–624, 2021.

[5] M. Kummini, D. Veer, The h-polynomial and the rook polynomial of some polyominoes, arXiv:2110.14905, preprint.

[6] A.A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, J. Algebra 357, 279–303, 2012.

[7] A. A. Qureshi, G. Rinaldo, F. Romeo, Hilbert series of parallelogram polyominoes, Res. Math. Sci. **9**, 28, 2022.

Om Prakash

Indian Institute of Technology Gandhinagar, India

On affine semigroups of maximal projective dimension

A submonoid of N^d is of maximal projective dimension (MPD) if the associated affine semigroup ring has the maximum possible projective dimension. Such submonoids have a nontrivial set of pseudo-Frobenius elements. We generalize the notion of symmetric semigroups, pseudo-symmetric semigroups, and row-factorization matrices for pseudo-Frobenius elements of numerical semigroups to the case of MPD-semigroups in N^d. Under suitable conditions, we prove that these semigroups satisfy the Extended Wilf's conjecture. We prove that the generic nature of the defining ideal of the associated semigroup ring of an MPD-semigroup implies the uniqueness of the row-factorization matrix for each pseudo-Frobenius element. Further, we give a description of pseudo-Frobenius elements and row-factorization matrices of gluing of MPD-semigroups. Joint work with Dr. Kriti Goel and Prof. Indranath Sengupta.

Masoomeh Rahimbeigi

Iran

Classes of cut ideals and their Betti numbers

We study monomial cut ideals associated to a graph G, which are a monomial analogue of toric cut ideals as introduced by Sturmfels and Sullivant. Primary decompositions, projective dimensions, and Castelnuovo-Mumford regularities are investigated if the graph can be decomposed as 0-clique sums and disjoint union of subgraphs. The total Betti numbers of a cycle are computed. Moreover, we classify all Freiman ideals among monomial cut ideals.

Kamalesh Saha

Indian Institute of Technology Gandhinagar, India

The v-number of monomial ideals

We show that the v-number of an arbitrary monomial ideal is bounded below by the v-number of its polarization and also, find a criteria for the equality. By showing the additivity of associated primes of monomial ideals, we obtain the additivity of the v-numbers for arbitrary monomial ideals. We prove that the v-number $\mathrm{v}(I(G))$ of the edge ideal $I(G)$, the induced matching number $\mathrm{im}(G)$ and the regularity $\mathrm{reg}(R/I(G))$ of a graph G, satisfy $\mathrm{v}(I(G)) \leq \mathrm{im}(G) \leq \mathrm{reg}(R/I(G))$, where G is either a bipartite graph, or a (C_4, C_5)-free vertex decomposable graph, or a whisker graph. Jaramillo et al., have proposed an open problem, whether $\mathrm{v}(I) \leq \mathrm{reg}(R/I) + 1$ for any square-free monomial ideal I. We show that $\mathrm{v}(I(G)) > \mathrm{reg}(R/I(G)) + 1$, for a disconnected graph G. We derive some inequalities of v-numbers which may be helpful to answer the above problem for the case of connected graphs. We connect $\mathrm{v}(I(G))$ with an invariant of the line graph $L(G)$ of G. For a simple connected graph G, we show that $\mathrm{reg}(R/I(G))$ can be arbitrarily larger than $\mathrm{v}(I(G))$. Also, we try to see how the v-number is related to the Cohen-Macaulay property of square-free monomial ideals.

Pranjal Srivastava

Indian Institute of Technology Gandhinagar, India

Projective closure of affine monomial curves

We introduce the notion of star gluing of numerical semigroups and show that arithmetically Cohen-Macaulay and Gorenstein properties of the projective closure are preserved under this gluing operation. We then give a condition on Gröbner basis of the defining ideal of an affine monomial curve which ensures that the Betti sequence of the affine curve is the same as the Betti sequence of its projective closure. We also study the effect of simple gluing on Betti sequences of the projective closure. Finally, we construct some numerical semigroups, using a gluing technique, such that the Cohen-Macaulay type of corresponding affine curve and its projective closure are both n. This is joint work with Prof. Indranath Sengupta and Dr Joydip Saha.

Meral Süer

Batman University, Turkey

Arf numerical semigroups and their gaps

In this talk, we examine the set of Arf numerical semigroups containing a given Arf numerical semigroup. For this, we use the gaps of the given Arf numerical semigroup. We present an algorithm that calculates all numerical semigroups containing the given Arf numerical semigroup.

Dharm Veer

Chennai Mathematical Institute, India

On h-polynomial and rook polynomial of polyominoes

Let $\mathcal{P}$ be a polyomino. Qureshi associated a finitely generated graded algebra $K[\mathcal{P}]$ over a field K to $\mathcal{P}$. Let $h(t) = (1 + h_1 t + h_2 t^2 + \cdots)/(1-t)^{\dim(K[\mathcal{P}])}$ be the Hilbert series of $K[\mathcal{P}]$. The rook polynomial $r_{\mathcal{P}}(t)$ of $\mathcal{P}$ is $\sum_{k\in\mathbb{N}} r_k t^k$ where for $k \in \mathbb{N}$, r_k is the number of configurations in $\mathcal{P}$ with k pairwise non-attacking rooks. Rinaldo and Romeo showed that if $\mathcal{P}$ is a simple thin polyomino, then $h(t) = r_{\mathcal{P}}(t)$ and conjectured that this property characterises thin polyominoes.

In this talk, we show that $h_2 < r_2$ when $\mathcal{P}$ is a non-thin convex polyomino such that its vertex set is a sublattice of $\mathbb{N}^2$. We also show that the Gorenstein rings associated to simple thin polyominoes satisfy the Charney-Davis conjecture. This is a joint work with Manoj Kummini.
