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# Orthogonal Matrices: Third Time Around

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This presentation reflects new ideas about a topic which has been discussed at previous ACA conferences. Orthogonal matrices $Q$ are defined by $QQ^t = dI$ where $Q^t$ is the transpose, and $I$ is the identity matrix with $d \neq 0$. Orthonormal matrices denote the special case $d = 1$. In addition to the important role they play in applications, such as the $QR$ decomposition, orthogonal matrices are also useful to instructors in Linear Algebra and multivariable calculus. For example, if an instructor wants to create a problem in which lines intersect at a given angle, then one way to do this is to create the problem in a simple configuration and then use an orthogonal matrix to transform the problem to a more complicated one. Thus, the lines $L_1 = r\langle 1, 0, 0\rangle, L_2 = s\langle 1, 1, 0\rangle, L_3 = \langle 2, 0, 0\rangle + t\langle 0, 1, 0\rangle$ form a simple triangle, but after multiplying by $Q$, they still form a triangle with the same angles, but are now shifted and rotated in space.

For pedagogical reasons, it is very helpful to students if $Q$ matrices have rational elements. In this talk, we shall discuss various ways in which orthogonal and orthonormal matrices with purely rational entries can be computed. Our aim is to create a repository of rational orthogonal matrices for instructors to use when creating examples and exercises. Access will be free and open.

The first approach utilises an exhaustive search. Since the columns of any rational orthonormal matrix must form a Pythagorean $n$-tuple, we start by generating the list of all primitive Pythagorean $n$-tuples where the entries are below a certain size. We then combine the tuples until we have found an orthonormal matrix. (Note that we only need to find $n - 1$ columns in this way; the $n$-th column is then uniquely determined and can be computed by other means.) The benefit of this method is that we gain tight control over the sizes of the matrix entries, which is helpful for generating and ordering our open database of orthogonal matrices. A repository is more useful if the entries are not randomly presented.

A second method is based on a result by Cayley [1]: If $A$ is a skew-symmetric rational matrix, then $(I - A)^{-1}(I + A)$ will be orthogonal; and all rational orthogonal matrices which do not have $1$ as an eigenvalue can be obtained in this way. In [2], Liebeck and Osborne have shown that every orthogonal matrix can be transformed into an orthogonal matrix for which $1$ is not an eigenvalue through multiplication of its rows by $\pm 1$. Hence, Cayley's formula can be used to obtain all orthogonal matrices. We analyse some interesting patterns which arise from the use of this method.

Finally, rational orthonormal matrices of higher dimension can be generated by composing smaller orthonormal matrices. For example, it is well known (see, e. g., [3]) that the Kronecker product of orthonormal matrices is itself orthonormal. Also, block diagonal matrices where the blocks are orthonormal will themselves be orthonormal. If we multiply these block diagonal matrices by random permutations, it becomes easy to generate orthonormal matrices with a predefined degree of sparseness.

**Keywords**
Orthogonal matrices, Pythagorean numbers, undergraduate education

**References**
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