Surfaces and their Duals

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This paper is dedicated to the memory of my close friend Eugenio Roanes-Lozano, a driving force of Applications of Computer Algebra.

The pictures were taken at the occasion of a private visit and stay of Eugenio in our home in Würmla.

Introduction

I have a large collection of papers and articles from various journals, conference proceedings, seminars and other publications.

From time to time I like to browse through my 10 or 11 thick folders. Then I remove one or the other paper which is out of time or find it interesting enough for treating it by means of Dynamic Geometry and/or Computer Algebra.

So, I came across two pages from a Maths&Stats Journal from 2002 where Dr Richard Morris presented *Dual Surfaces*.

I had the idea to transfer the ideas presented by a short text and some pictures of low quality to my preferred CAS tools: DERIVE and TI-NspireCAS. This was an exciting task, although remaining on the level of Upper Secondary school – or just a little bit above.

This was Dr Morris' article:

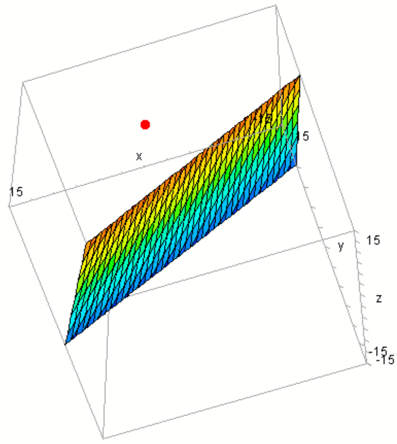
Dr Richard Morris, Liverpool University

The pictures in this issue of Maths&Stats are all of various surfaces and their duals. The dual of a surface is the set of planes tangent to the surface. Any plane, a x + b y + c z = d, can be thought of as a point (a,b,c,d) in RP^2. Hence the dual of a surface forms a surface in RP^2. Each point on the surface will have a tangent plane and hence give a point on the dual. What is actually shown here are projections of the duals into R^3 using the map (a,b,c,d)->(a/c, b/c, d/c).

The pictures here have been produced using the Liverpool Surface Package and Geomview. The Liverpool Surface Package is a set of programs which can generate a variety of mathematical curves and surface, defined either by a parametrization or by implicit algebraic equations. Other options such as performing mapping from R^3 to R^3 and constructing duals are also possible. Geomview is an interactive 3D geometry viewer, written at the Geometry Center, University of Minneapolis, Minnasota. Geomview is public domain software available via anonymous ftp from geom.edu.umn. Enquiries about the Liverpool Surface Package should be e-mailed to rmorris@liverpool.ac.uk. Both packages run on Silicon Graphics Irises.

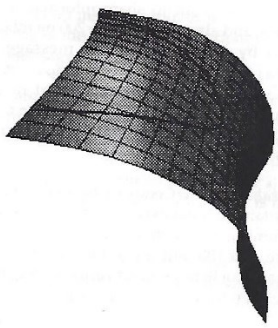
3·x + 7·y + z = 12

[3, 7, 12]



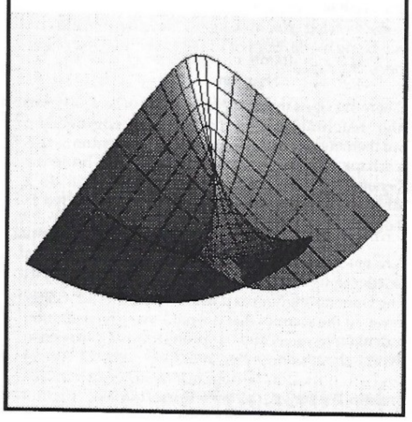
A plane and its dual point

$z = x^2 + y^3$



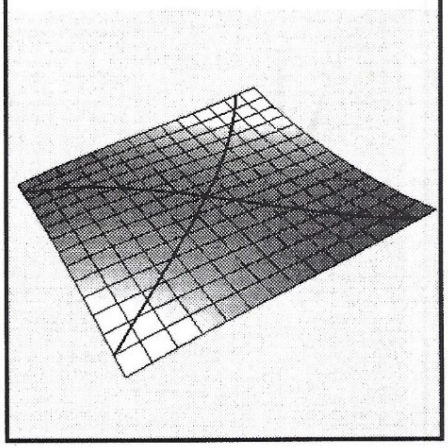
1. A surface with a parabolic line running along it - see page 6 for its dual.

2. The dual of the surface on page 1

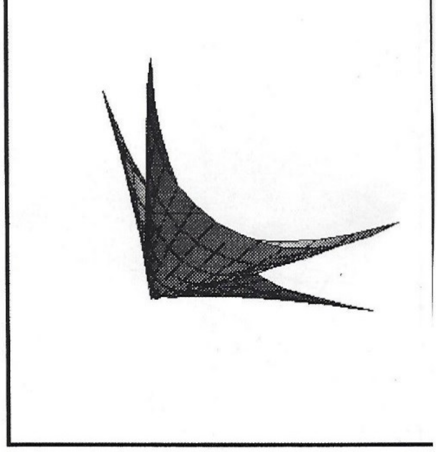


$z = x^3 + xy^2$ hyperbolic cubic form

3. A flat hyperbolic umbilic surface - see over for its dual

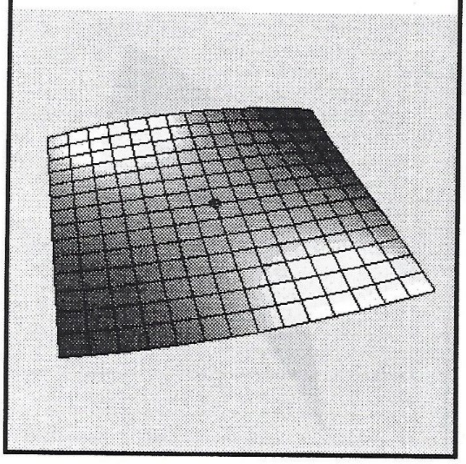


4. Dual of surface on previous page - see page 23

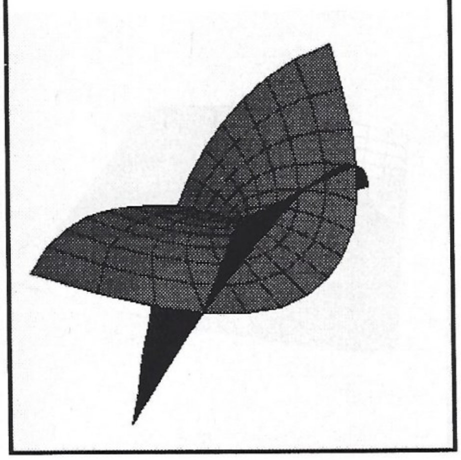


$z = x^3 - xy^2$ (elliptical cubic form)

7. A flat elliptical umbilic surface - see next page for dual



8. Dual of surface on previous page - see page 35



I was confused, because Dr Morris writes about RP^2 in his short article, but then about RP^4 in his extended paper – which seems to have more sense.

I will perform this mapping using my favourite Computer Algebra systems: DERIVE and TI-NspireCAS.

I must admit that I don't know about RP^2 (Real Projective Plane). I wanted to download the program mentioned *Geomview*. It is free but it runs only under UNIX.

https://journals.gre.ac.uk/index.php/msor/issue/archive

Searching for the original article from 2002 I found a much more extended paper on the same issue, also from Dr Morris and also from 2002 at

www.singsurf.org/papers/dual/dual.pdf but unfortunately, the server cannot be found any more.

What do we need? The Tangent Plane of a Surface

Surface	Tangent Plane
explicit – implicit form $z = f(x,y)$ $F(x,y,z) = f(x,y) - z = 0$	$grad(F(u,v,w)) \cdot \begin{pmatrix} x-u \\ y-v \\ z-w \end{pmatrix} = 0$
parameter form $p(u,v) = \begin{pmatrix} px(u,v) \\ py(u,v) \\ pz(u,v) \end{pmatrix}$ $f(x,y) \to p(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$	$\begin{pmatrix} x-px(u,v) \\ y-py(u,v) \\ z-pz(u,v) \end{pmatrix} \cdot \left(\frac{d(p(u,v))}{du} \times \frac{d(p(u,v))}{dv} \right) = 0$

Morris' first example stepwise:

```
#1:   tf_expl(f) := SUBST(GRAD(f - z), [x, y], [u, v])·([x, y, z] -
        SUBST([x, y, f], [x, y], [u, v])) = 0
```

#2: $f1 := x^2 + y^3$

#3: $tf\_expl(f1) = (2 \cdot u \cdot x + 3 \cdot v^2 \cdot y - z - u^2 - 2 \cdot v^3 = 0)$

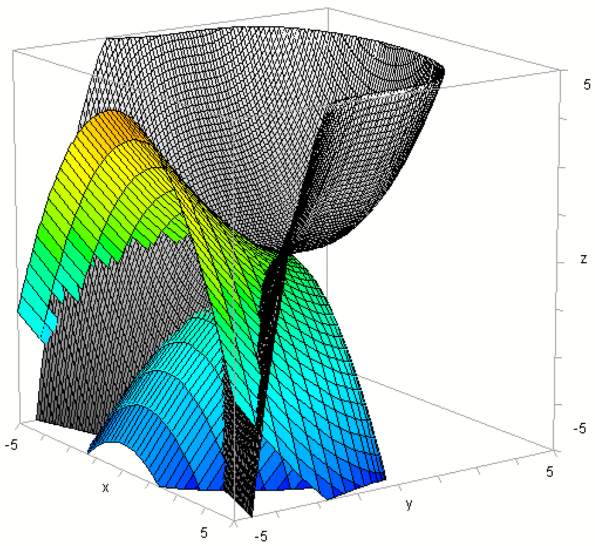
#4: $2 \cdot u \cdot x + 3 \cdot v^2 \cdot y - z - u^2 - 2 \cdot v^3 = 0$

(a/c,b/c,d/c)

#5: $\left[\frac{2 \cdot u}{-1}, \frac{3 \cdot v^2}{-1}, \frac{u^2 + 2 \cdot v^3}{-1} \right]$

Parameter form of the Dual:

#6: $\left[-2 \cdot u, -3 \cdot v^2, -u^2 - 2 \cdot v^3 \right]$



Origin in Gray Scale, Dual in Rainbow

Later we will come back to the cuspidal edge of the dual.

For Morris's second example we will work with the parameter form:

Instead of $z(x,y)=x^3+xy^2$ we take $p(u,v)=\begin{pmatrix} u \\ v \\ u^3+u\,v^2 \end{pmatrix}$.

#7: $f2 := x^3 + x \cdot y^2$

#8: $p2 := [u, v, u^3 + u \cdot v^2]$

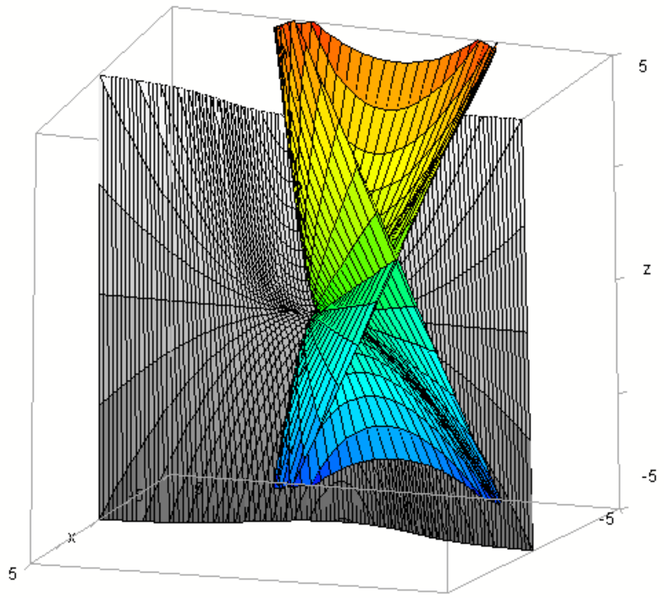
#9: $tp(p) := \text{EXPAND}\left([x - p_1, y - p_2, z - p_3] \cdot \left(\left(\frac{d}{du} p\right) \times \frac{d}{dv} p\right)\right) = 0$

#10: tp(p2)

#11: $2 \cdot u^3 - 3 \cdot u^2 \cdot x + 2 \cdot u \cdot v^2 - 2 \cdot u \cdot v \cdot y - v^2 \cdot x + z = 0$

#12: $- 3 \cdot u^2 \cdot x - 2 \cdot u \cdot v \cdot y - v^2 \cdot x + z = - 2 \cdot u^3 - 2 \cdot u \cdot v^2$

#13: $[- 3 \cdot u^2 - v^2, - 2 \cdot u \cdot v, - 2 \cdot u^3 - 2 \cdot u \cdot v^2]$



We collect the steps in two – very – short programs (functions) for parameter and explicit form:

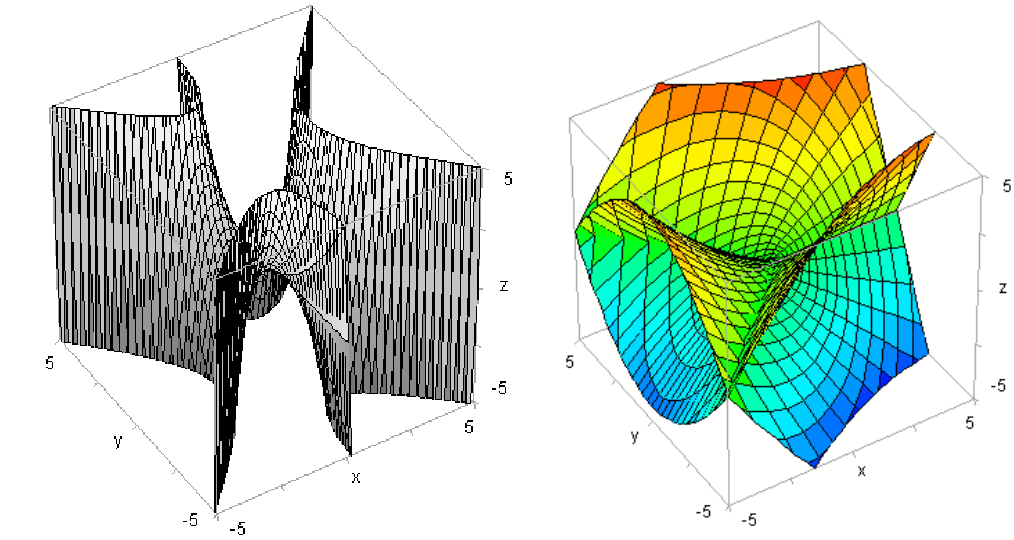
```
          dualp(p, dt, d) :=
            Prog
#14:          dt := [x - p↓1, y - p↓2, z - p↓3]·(∂(p, u) × ∂(p, v))
              d := - SUBST(dt, [x, y, z], [0, 0, 0])
              [(GRAD(dt))↓1, (GRAD(dt))↓2, d]/(GRAD(dt))↓3

          dualf(f, p, dt, d) :=
            Prog
#15:          p := [u, v, SUBST(f, [x, y], [u, v])]
              dualp(p)
```

Let’s test them with Morris’ third example:

#16: $f3 := x^3 - x \cdot y^2$

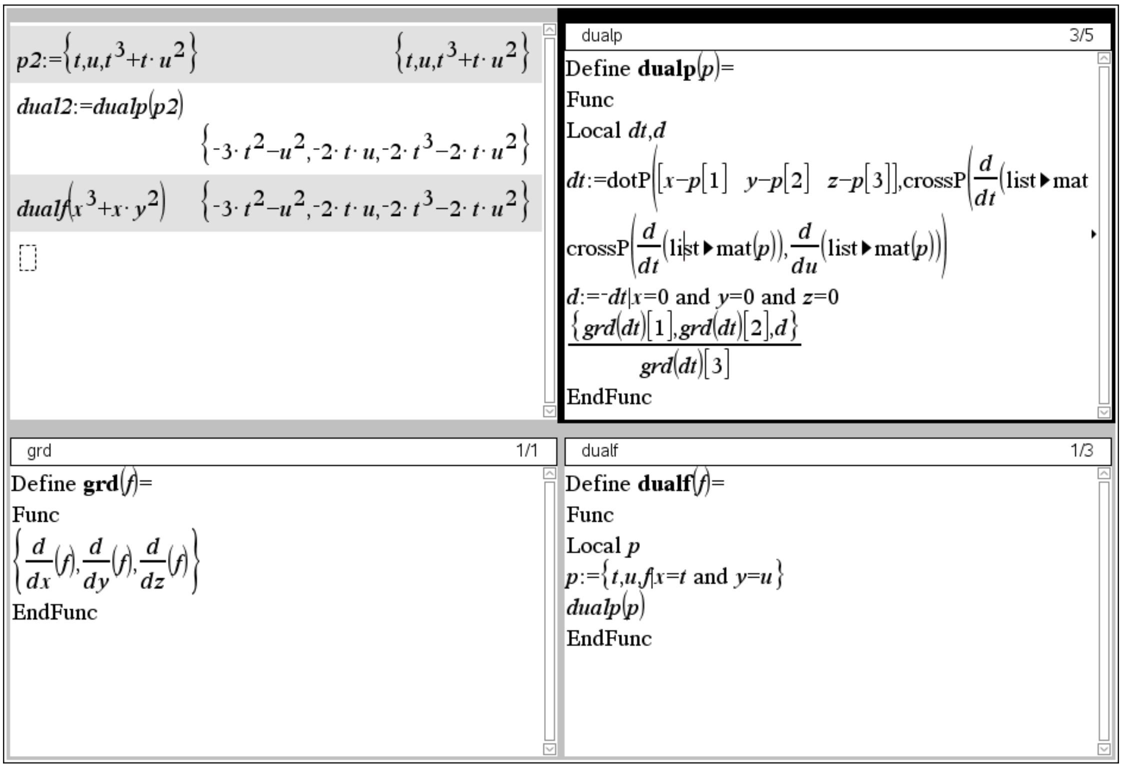
#17: $\text{dualf}(f3) = \left[v^2 - 3 \cdot u^2,\ 2 \cdot u \cdot v,\ 2 \cdot u \cdot (v^2 - u^2)\right]$

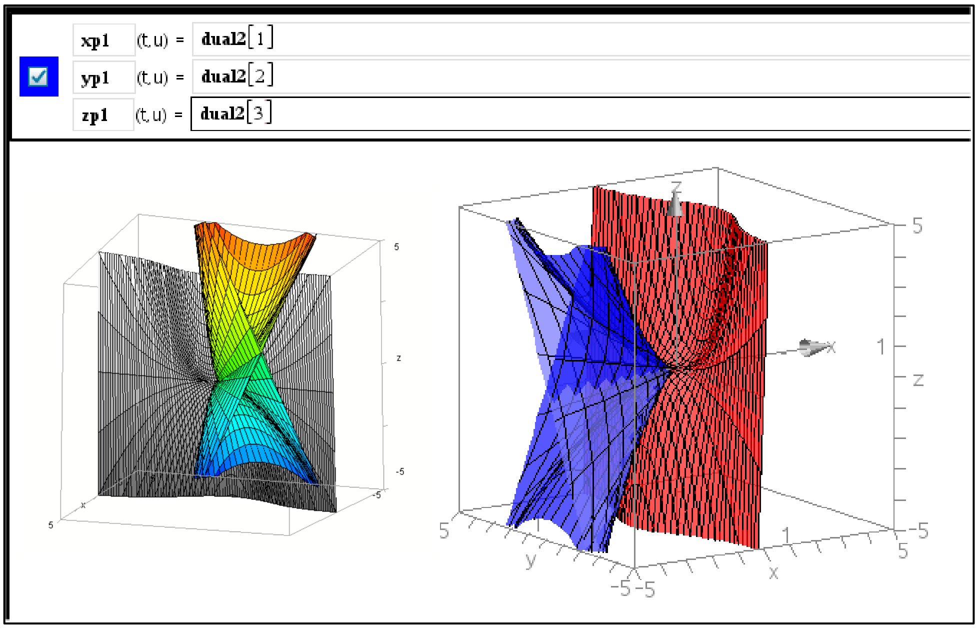
#18: $\text{dualp}\left(\left[u,\ v,\ u^3 - u \cdot v^2\right]\right) = \left[v^2 - 3 \cdot u^2,\ 2 \cdot u \cdot v,\ 2 \cdot u \cdot (v^2 - u^2)\right]$



I will come back to Morris’ examples at the end of the paper.

Have a look how TI-NspireCAS performs with the tasks:





Compare with the DERIVE plot (left)!

Until here this is the realization of the short article in the journal.

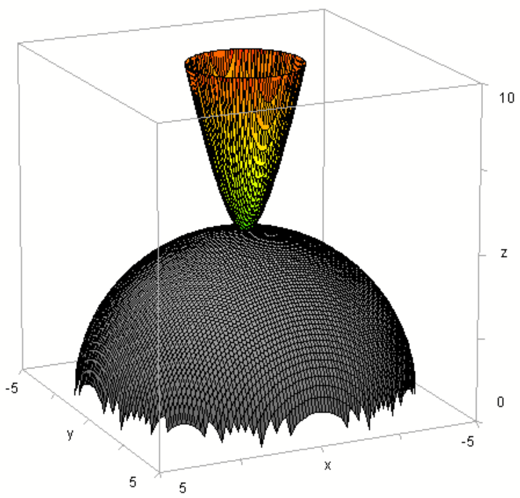
We can start experimenting with surfaces of our choice.

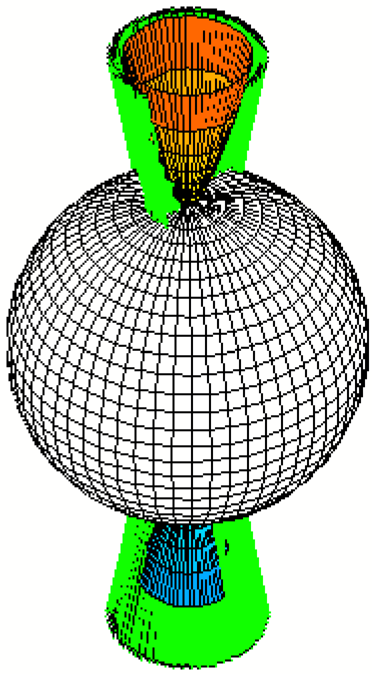
I begin with a simple sphere:

#20: $\text{dualf}\left(\sqrt{25 - x^2 - y^2}\right)$

#21: $\left[\dfrac{u}{\sqrt{- u^2 - v^2 + 25}}, \dfrac{v}{\sqrt{- u^2 - v^2 + 25}}, \dfrac{25}{\sqrt{- u^2 - v^2 + 25}}\right]$

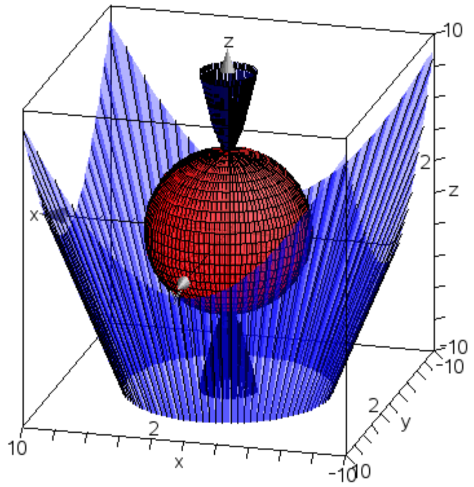
Below the dual surface of the function and right the dual surface of the complete sphere (parameter form).





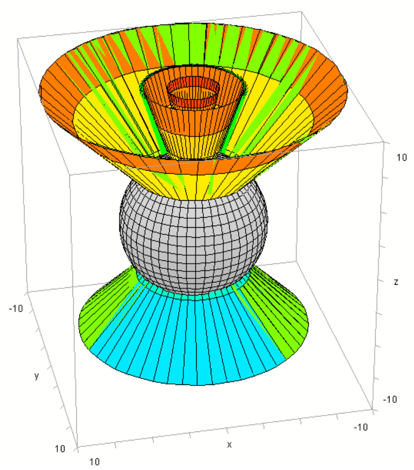
Let’s try the parameter form on the Nspire:

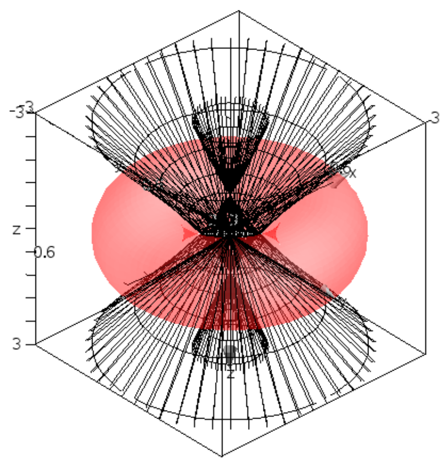
$$dualsph:=dualp(\{5\cdot \sin(t)\cdot \cos(u),5\cdot \sin(t)\cdot \sin(u),5\cdot \cos(t)\}) \blacktriangleright \left\{\tan(t)\cdot \cos(u),\tan(t)\cdot \sin(u),\frac{5}{\cos(t)}\right\}$$



Nice “crown with four spikes”.

Torus





$$\mathbf{torus} \blacktriangleright \{\cos(t)\cdot(\cos(u)+2),\sin(t)\cdot(\cos(u)+2),\sin(u)\}$$

$$\mathbf{dualp(torus)} \blacktriangleright \left\{\frac{\cos(t)}{\tan(u)},\frac{\sin(t)}{\tan(u)},\frac{2\cdot \cos(u)+1}{\sin(u)}\right\}$$

I compare the 3rd component of the dual given by DERIVE (left) and by TI-NspireCAS (right):

$$\frac{\mathrm{COS}(u)^3\cdot \mathrm{COS}(v)^2 + 2\cdot \mathrm{COS}(u)^2 + \mathrm{COS}(u)\cdot(\mathrm{SIN}(u)^2\cdot \mathrm{COS}(v)^2 + \mathrm{SIN}(v)^2 + 4) + 2}{\mathrm{SIN}(u)\cdot(\mathrm{COS}(u) + 2)} - \frac{2\cdot \mathrm{COS}(u) + 1}{\mathrm{SIN}(u)}$$

Simplifying the difference of both expressions gives 0 – both expressions are identical.

0

Trigonometry := Collect

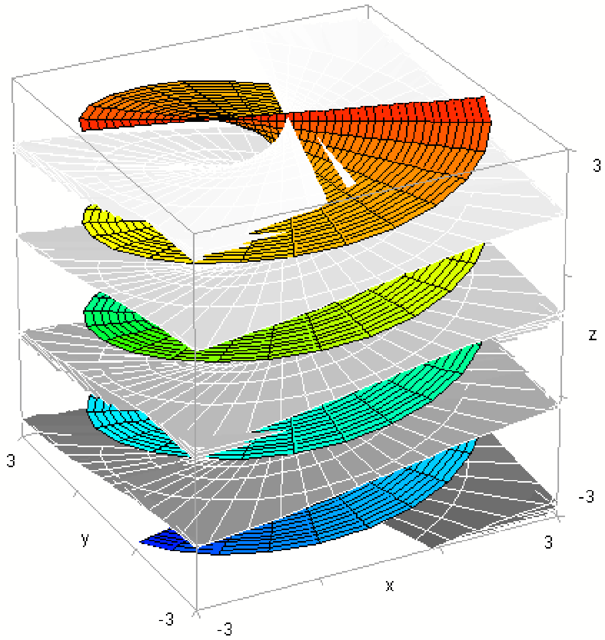
$$\frac{2\cdot(\text{COS}(2\cdot u) + 5\cdot\text{COS}(u) + 3)}{\text{SIN}(2\cdot u) + 4\cdot\text{SIN}(u)} - \frac{2\cdot\text{COS}(u) + 1}{\text{SIN}(u)}$$

Trigonometry := Expand

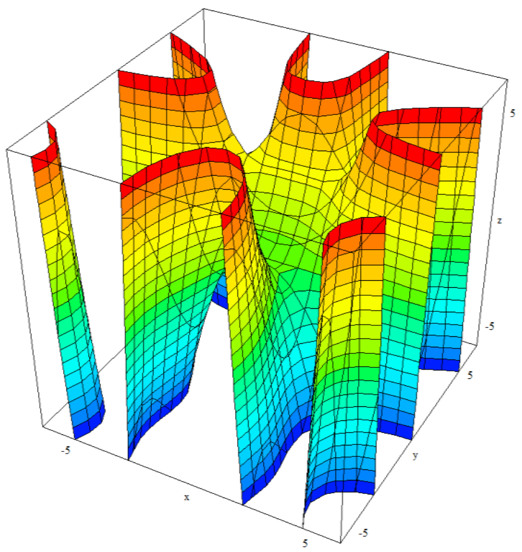
$$\left(\frac{2\cdot\text{COS}(u)}{\text{SIN}(u)} + \frac{1}{\text{SIN}(u)}\right) - \frac{2\cdot\text{COS}(u) + 1}{\text{SIN}(u)}$$

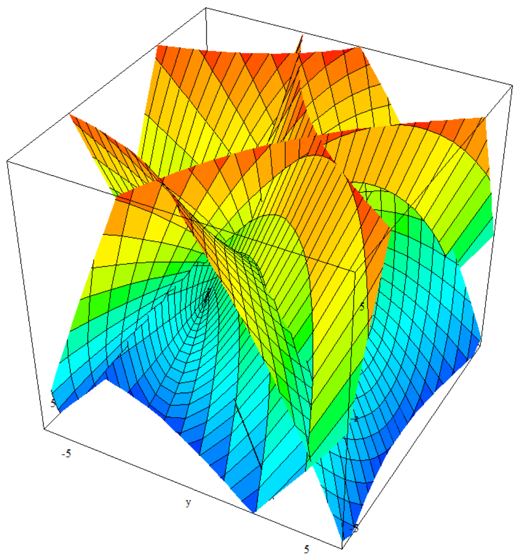
Short manipulation with trig expressions shows the identity, too. (Let's try manually ☺)

Helicoid: [3·v·COS(u), 3·v·SIN(u), 0.5·u]



Finally, I'd like to find the duals of surfaces of my phantasy: z=x^2·SIN(y) + y^2·COS(x)





Graphs are produced using David Parker's DPGraph.

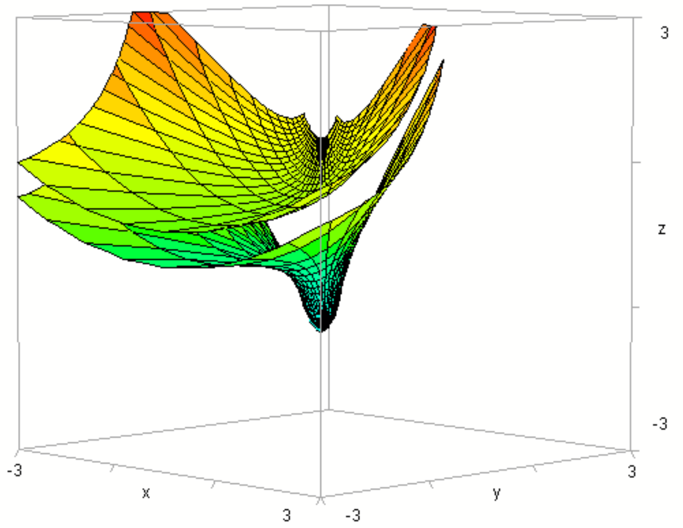
Another dual?

Dr. Morris: "... *using the map (a,b,c,d) → (a/c,b/c,d/c)*". Why not $(a,b,c,d) \to (a/b,c/b,d/b)$?

We will apply this map on the first example:

$$2\cdot u\cdot x + 3\cdot v^2\cdot y - z - u^2 - 2\cdot v^3 = 0$$

$$\left[\frac{2\cdot u}{3\cdot v^2}, -\frac{1}{3\cdot v^2}, \frac{u^2 + 2\cdot v^3}{3\cdot v^2}\right]$$

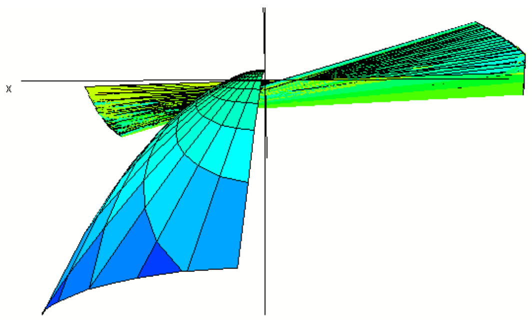


Looks quite different!

Here are the other duals:

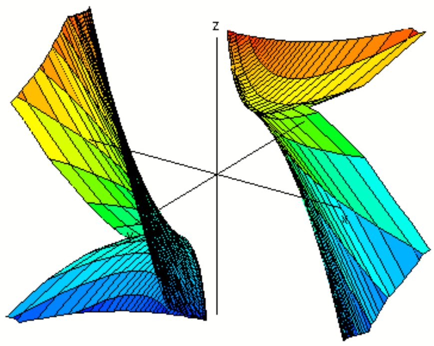
$(a/d,b/d,c/d)$

$$\left[\frac{2*u}{u^2 + 2*v^3}, \frac{3*v^2}{u^2 + 2*v^3}, -\frac{1}{u^2 + 2*v^3}\right]$$



$(b/a,c/a,d/a)$

$$\left[\frac{3\cdot v^2}{2\cdot u}, -\frac{1}{2\cdot u}, \frac{u^2 + 2\cdot v^3}{2\cdot u}\right]$$



I will remain at this first easy surface and follow the next idea:

I intend to find the mappings of parameter curves.

#63 and #64 are the *v*-lines on surface (#61) and its dual (#62) (blue), #65 and #66 are the *u*-lines (red).

#61: $\left[u,\ v,\ u^2 + v^3\right]$

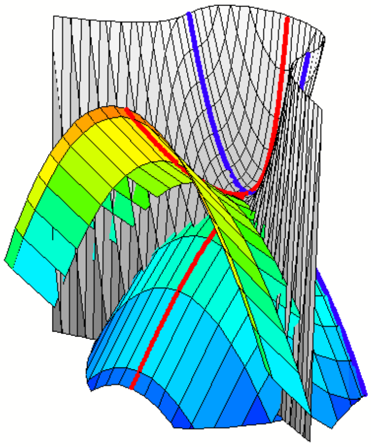
#62: $\left[-2\cdot u,\ -3\cdot v^2,\ -u^2 - 2\cdot v^3\right]$

#63: VECTOR$\left(\left[\left[0,\ v,\ v^3\right]\right],\ v,\ -5,\ 5,\ 0.01\right)$

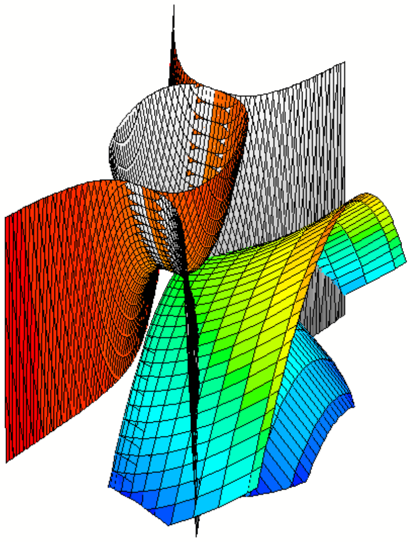
#64: VECTOR$\left(\left[\left[0,\ -3\cdot v^2,\ 0 - 2\cdot v^3\right]\right],\ v,\ -5,\ 5,\ 0.01\right)$

#65: VECTOR$\left(\left[\left[u,\ 0,\ u^2\right]\right],\ u,\ -5,\ 5,\ 0.01\right)$

#66: VECTOR$\left(\left[\left[-2\cdot u,\ 0,\ -u^2\right]\right],\ u,\ -5,\ 5,\ 0.01\right)$



How does the **dual of the dual** of a surface look like?



#74: $f1 = x^2 + y^3$

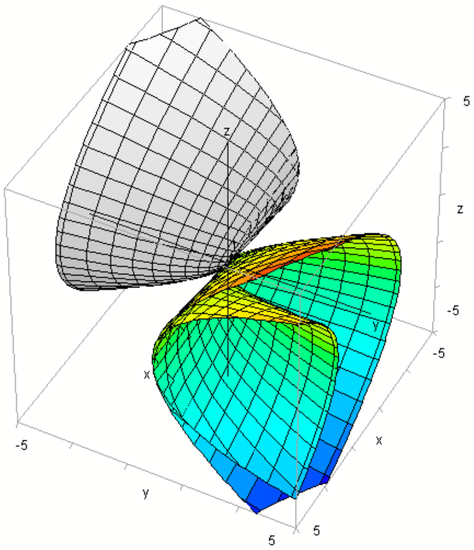
#75: $\mathrm{dualf}(f1) = [-2 \cdot u, -3 \cdot v^2, -u^2 - 2 \cdot v^3]$

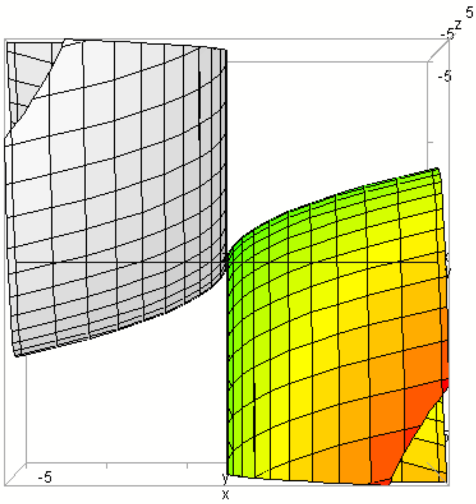
#76: $\mathrm{dualp}(\mathrm{dualf}(f1)) = [-u, -v, u^2 + v^3]$

#80: $[u^2 - v, v^2, u \cdot v]$

#81: $\mathrm{dualp}(\mathrm{dualp}([u^2 - v, v^2, u \cdot v]))$

#82: $[v - u^2, -v^2, u \cdot v]$





It seems to be that the dual of the dual results in a surface symmetric wrt the *z*-axis to the origin figure? Can we prove this?

Barack Obama: “Yes, we can!”

```
[f(u, v) :=, h(u, v) :=, g(u, v) :=]
```

$$\text{dualp}([g(u,v),\ h(u,v),\ f(u,v)]) = \left[\frac{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}f(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}f(u,v)}{\left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v) - \left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v)}, \right.$$

$$\frac{\left(\frac{d}{dv}g(u,v)\right)\cdot\frac{d}{du}f(u,v) - \left(\frac{d}{du}g(u,v)\right)\cdot\frac{d}{dv}f(u,v)}{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v)},$$

$$\frac{\left(\frac{d}{du}f(u,v)\right)\cdot\left(h(u,v)\cdot\frac{d}{dv}g(u,v) - g(u,v)\cdot\frac{d}{dv}h(u,v)\right) + \left(\frac{d}{dv}f(u,v)\right)\cdot\left(g(u,v)\cdot\frac{d}{du}h(u,v) - h(u,v)\cdot\frac{d\sim}{du\sim}\sim\right.}{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d\sim}{d\sim}\sim}$$

$$\left. \frac{g(u,v)\Big) + f(u,v)\cdot\left(\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v)\right)}{-\,g(u,v)\atop v} \right]$$

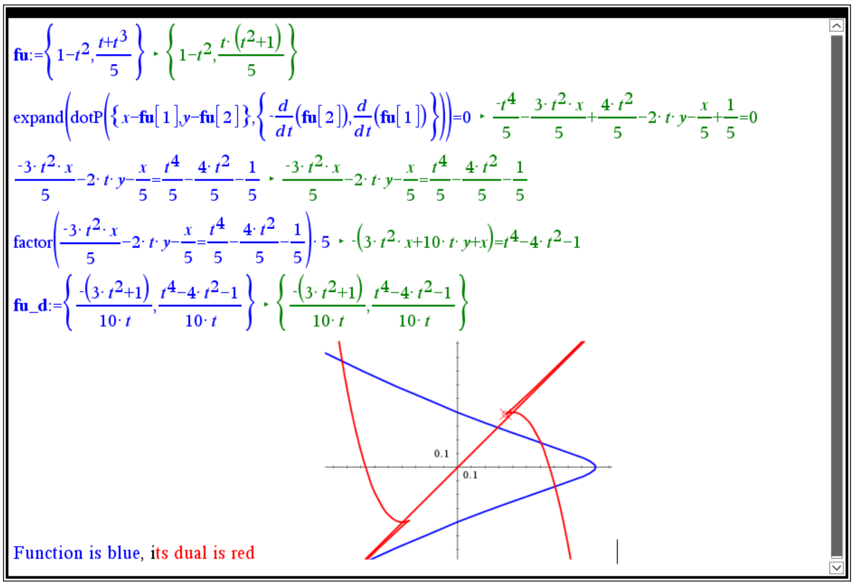
Pretty bulky expression, isn’t it? Give it a try:

```
dualp(dualp([g(u, v), h(u, v), f(u, v)])) = [– g(u, v), – h(u, v), f(u, v)]
```

Usually we raise a problem from the second dimension up to the third one – from plane to space. This time I do it the other way. I make a step down:

I map all tangents of a curve in the plane on points: $a\,x + b\,y = c \rightarrow \left(\frac{a}{b}, \frac{c}{b}\right)$.

I assume that the function to be mapped is given in parameter form $f(u) = (x(u), y(u))$.



The straight red line seems to be an asymptote?

Inspecting the red graph, we cannot avoid the next question: There are two cusps. Where are they? What are their origins on the given blue curve?

At first, we try to find the singularities and then we track them back to the blue curve:

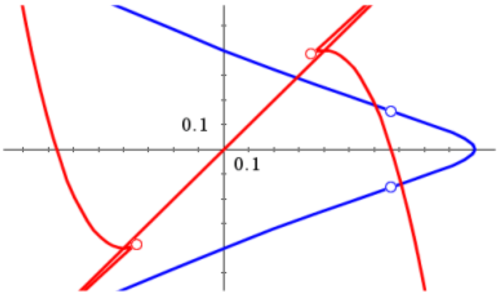
$$\text{solve}\left(\left\{\frac{d}{dt}(\mathbf{fu\_d}[1])=0,\frac{d}{dt}(\mathbf{fu\_d}[2])=0\right\},t\right) \blacktriangleright t=\frac{-\sqrt{3}}{3} \text{ or } t=\frac{\sqrt{3}}{3}$$

$$\mathbf{fu\_d}|t=\left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\} \blacktriangleright \begin{bmatrix}\frac{\sqrt{3}}{5} & \frac{-\sqrt{3}}{5}\\ \frac{2\cdot\sqrt{3}}{9} & \frac{-2\cdot\sqrt{3}}{9}\end{bmatrix}$$

the cusps on the dual curve ...

$$\mathbf{fu}|t=\left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\} \blacktriangleright \begin{bmatrix}\frac{2}{3} & \frac{2}{3}\\ \frac{-4\cdot\sqrt{3}}{45} & \frac{4\cdot\sqrt{3}}{45}\end{bmatrix}$$

... and where they come from.



Can it be, that the cusps are the mappings of the inflection points?

The points of inflection are where the curvature changes its sign.
So we calculate the zeros of the curvature of a curve given in parameter form -
- we need the zeros of the numerator:

$$\text{zeros}\left(\frac{d}{dt}(\mathbf{fu}[1])\cdot\frac{d^2}{dt^2}(\mathbf{fu}[2])-\frac{d^2}{dt^2}(\mathbf{fu}[1])\cdot\frac{d}{dt}(\mathbf{fu}[2]),t\right) \blacktriangleright \left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\}$$

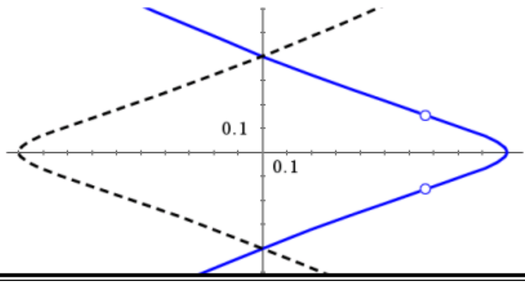
The dual points of the inflection points are the cusps.

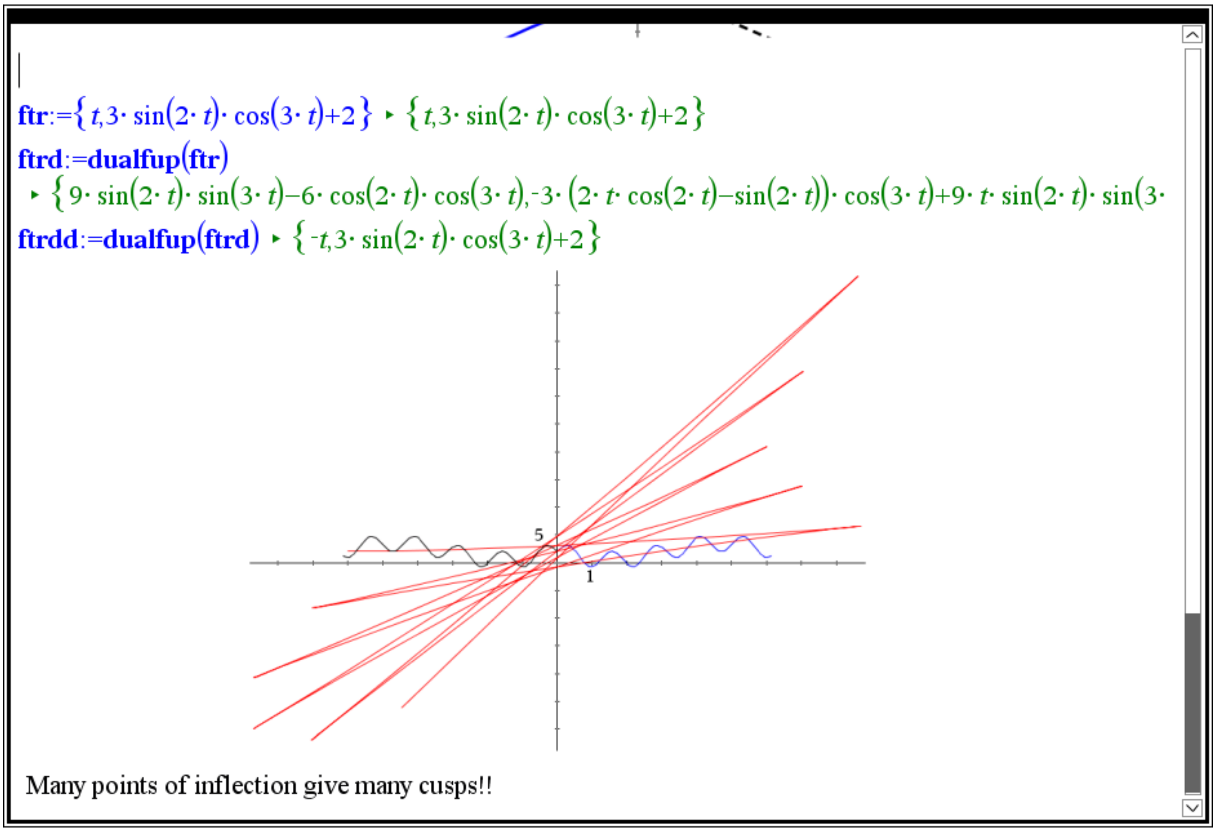
Last question:

Do you guess how the dual of the dual will look like?

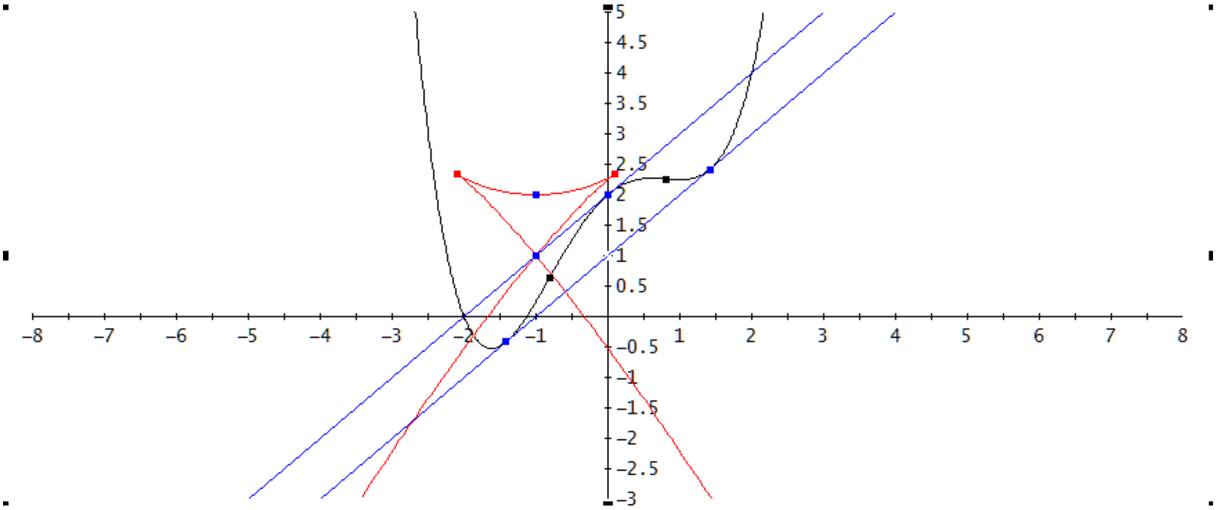
You are right:

$$\mathbf{fudd}:=\mathbf{dualfup}(\mathbf{dualfup}(\mathbf{fu})) \blacktriangleright \left\{t^2-1,\frac{t\cdot(t^2+1)}{5}\right\}$$





A quartic together with its dual:



Other investigations are pressing:

Function – Dual function

Inflection point – Cusp

Double tangent – Self intersection

I investigate a polynomial of order five together with its "dual":

Double tangents on a quintic …

$$z(x) := \frac{x^5 + 3\cdot x^4 - 11\cdot x^3 - 27\cdot x^2 + 10\cdot x + 64}{20}$$

$$z'(a) - z'(b) = 0$$

$$\text{FACTOR}(z(b) - z(a) - z'(a)\cdot(b - a) = 0)$$

$$eq1 := 5\cdot a^3 + a^2\cdot(5\cdot b + 12) + a\cdot(5\cdot b^2 + 12\cdot b - 33) + 5\cdot b^3 + 12\cdot b^2 - 33\cdot b - 54$$

$$eq2 := 4\cdot a^3 + a^2\cdot(3\cdot b + 9) + a\cdot(2\cdot b^2 + 6\cdot b - 22) + b^3 + 3\cdot b^2 - 11\cdot b - 27$$

GROEBNER_BASIS([eq1, eq2], [a, b])

$$\text{NSOLUTIONS}(500\cdot b^9 + 2700\cdot b^8 - 3740\cdot b^7 - 35004\cdot b^6 - 1837\cdot b^5 + 149937\cdot b^4 + 53017\cdot b^3 - 217071\cdot b^2 - 69909\cdot b + 38367,\ b)$$

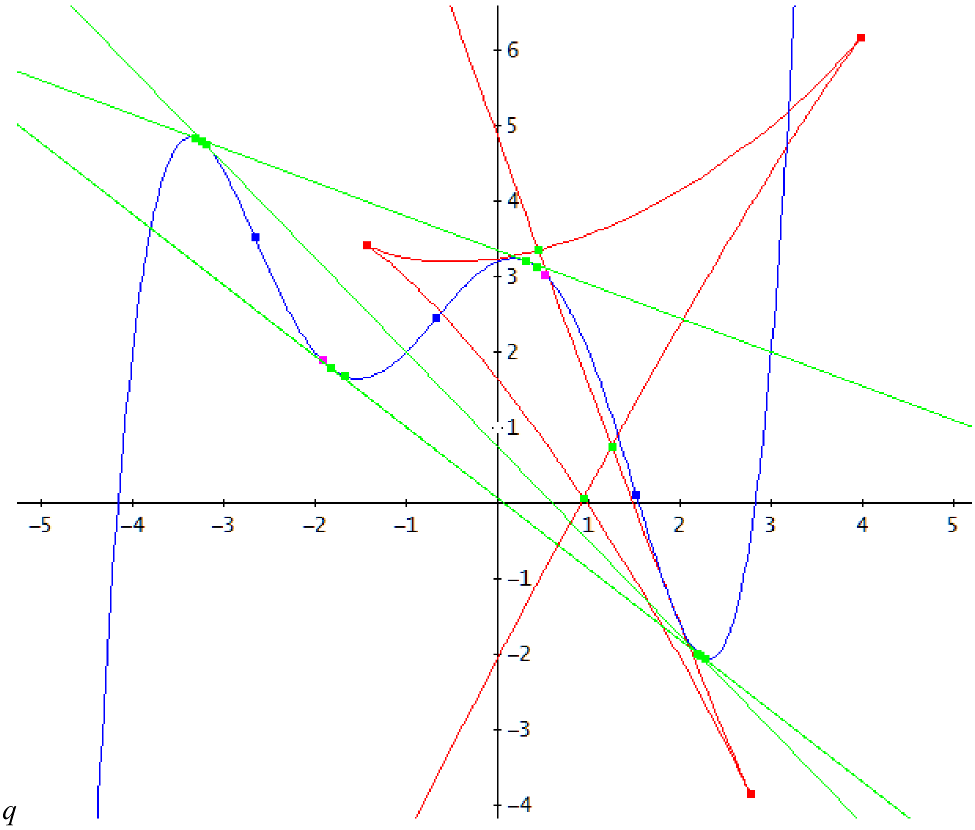
[2.224845673, -3.302775637, 2.192582403, -3.192582403, -1.824845673, 1.524977841, 0.3027756377, -0.6658185230, -2.659159318]

The Cusps

-1.438273175	3.411523680
3.974701467	6.157388302
2.769371708	-3.845895982

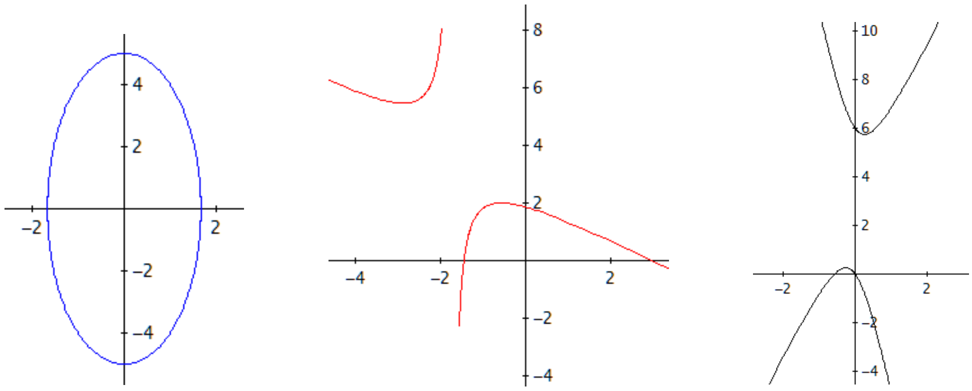
The Self Intersection points

0.9413000011	0.06811600289
0.45	3.35
1.25	0.75



Note: It can happen that you have interested and curious students. So, they might ask: “What are the duals of conics?”

Present the following graphs and invite them to guess from which conic they are the dual?



If necessary, provide a hint: points with vertical tangents are mapped to points of infinity!
(I must admit that I am not able to find the equation of the asymptotes of the duals. Any idea?)

(Solution: hyperbola (3 sinh(u), ±5 cosh(u)), parabola $((u^2 + 2\sqrt{3}u)/4, (u^2\sqrt{3} + 2u+8)/4$,
ellipse (5 cos(u) – 2, 3 sin(u) + 3)).

This is a nice calculation:

hyperbola1 := [3·SINH(u), 5·COSH(u)]

$$\text{dualfup(hyperbola1)} = \left[\frac{5\cdot(1 - e^{2\cdot u})}{3\cdot(e^{2\cdot u} + 1)}, \frac{10\cdot e^{u}}{e^{2\cdot u} + 1}\right]$$

Is this really an ellipse?

$$\left[\frac{5\cdot(e^{2\cdot u} - 1)}{3\cdot(e^{2\cdot u} + 1)}, -\frac{10\cdot e^{u}}{e^{2\cdot u} + 1}\right]$$

$$\left[\frac{5\cdot(e^{2\cdot \mathrm{LOG}(t)} - 1)}{3\cdot(e^{2\cdot \mathrm{LOG}(t)} + 1)}, -\frac{10\cdot e^{\mathrm{LOG}(t)}}{e^{2\cdot \mathrm{LOG}(t)} + 1}\right] = \left[\frac{5\cdot(t^2 - 1)}{3\cdot(t^2 + 1)}, -\frac{10\cdot t}{t^2 + 1}\right]$$

$$\mathrm{SOLVE}\left(x = \frac{5\cdot(t^2 - 1)}{3\cdot(t^2 + 1)}, t\right) = \left(t = -\frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}} \vee t = \frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}}\right)$$

$$\mathrm{SUBST}\left(y = -\frac{10\cdot t}{t^2 + 1}, t, -\frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}}\right)$$

$$y = -\sqrt{(3\cdot x - 5)}\cdot\sqrt{(-3\cdot x - 5)}$$

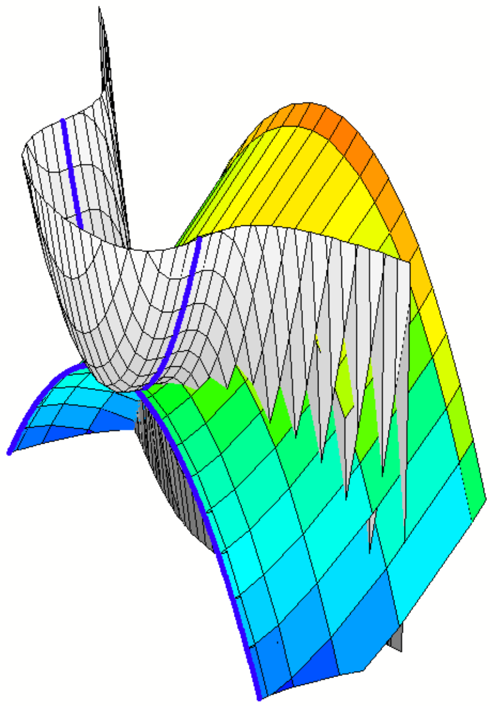
$$(y = -\sqrt{(3\cdot x - 5)}\cdot\sqrt{(-3\cdot x - 5)})^2 = (y^2 = 25 - 9\cdot x^2)$$

Back to the Surfaces!

The inflection points and the cusps reminded me on the parameter curves of the first surface.

The mapping of the parameter curve on the given surface is the sharp – cuspidal – edge of its dual.

I get the idea that the origin curve is a locus of something like inflection points. It looks like a curve on a "terrace in the landscape" …



It is not long ago that I found a more extended Morris paper on the same subject *Visualising Duals of Surfaces*:

www.singsurf.org/papers/dual/dual.pdf.

(Unfortunately, this website seems to be not available any longer.)

"*If x lies on a parabolic line on S then the corresponding point on the dual, x*, lies on a cuspidal edge on S**."

So, what is a *parabolic line*?

Points with *Gauss Curvature* = 0 are *parabolic points* and the set of parabolic points form a *parabolic line*. The respective maps are *cuspidal points* and *cuspidal edges*.

Next function gives the Gauss curvature of function $p(u,v)$:

I found the Gaussian Curvature and how to calculate it in Alfred Gray's "*Modern Differential Geometry* …"

This is beyond Upper Secondary level, but it might be a good exercise providing for the students A. Gray's text and ask them to produce a working function (DERIVE, TI-NspireCAS, wxMaxima, …)

```
gauss_curv(p, u, v, e_, f_, g_, e, f, g, h) :=
  Prog
    e_ := ∂(p, u)·∂(p, u)
    f_ := ∂(p, u)·∂(p, v)
    g_ := ∂(p, v)·∂(p, v)
    h := √(e_·g_ - f_^2)
    e := DET([∂(p, u, 2), ∂(p, u), ∂(p, v)])/h
    f := DET([∂(∂(p, u), v), ∂(p, u), ∂(p, v)])/h
    g := DET([∂(p, v, 2), ∂(p, u), ∂(p, v)])/h
    (e·g - f^2)/h^2
```

$$\text{gauss\_curv}([u,\ v,\ u^2 + v^3]) = \frac{12\cdot v}{(4\cdot u^2 + 9\cdot v^4 + 1)^2}$$

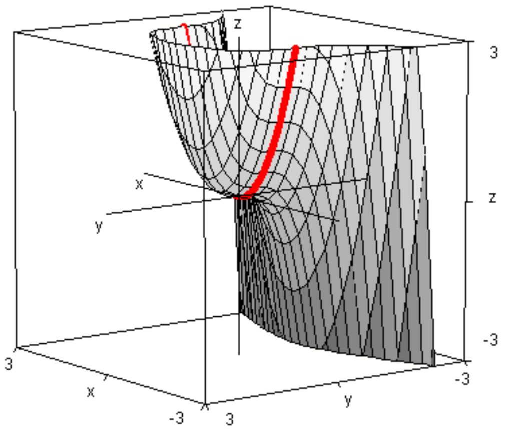
$v = 0$ gives the parabolic line on the given surface and the cuspidal edge on its dual.

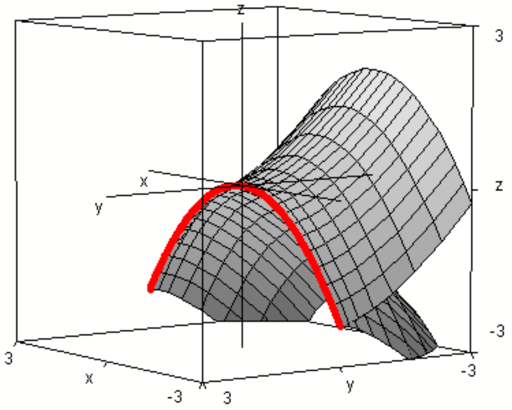
$$\text{VECTOR}([[u,\ 0,\ u^2]],\ u,\ -5,\ 5,\ 0.01)$$

The dual is:

$$\left[-2\cdot u,\ -3\cdot v^{2},\ -u^{2}-2\cdot v^{3}\right]$$

$$\mathrm{VECTOR}\left(\left[\left[-2\cdot u,\ 0,\ -u^{2}\right]\right],\ u,\ -5,\ 5,\ 0.01\right)$$





I come back to function $f2 = x^3 + x\,y^2$ in order to confirm my findings:

$$\mathrm{surf} := \left[u,\ v,\ u^{3}+u\cdot v^{2}\right]$$

$$\mathrm{gauss\_curv(surf)} = \frac{4\cdot(3\cdot u^{2}-v^{2})}{(9\cdot u^{4}+10\cdot u^{2}\cdot v^{2}+v^{4}+1)^{2}}$$

$$3u^2 - v^2 = 0$$

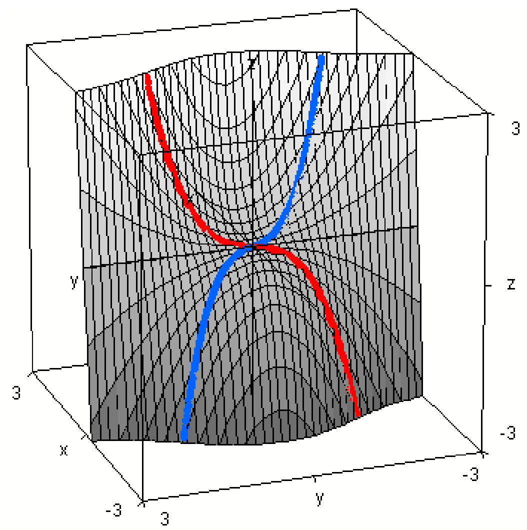
$$v = \pm u\sqrt{3}$$

$$\mathrm{SUBST}\left(\left[u,\ v,\ u^{3}+u\cdot v^{2}\right],\ v,\ u\cdot\sqrt{3}\right) = \left[u,\ \sqrt{3}\cdot u,\ 4\cdot u^{3}\right]$$

$$\mathrm{SUBST}\left(\left[u,\ v,\ u^{3}+u\cdot v^{2}\right],\ v,\ -u\cdot\sqrt{3}\right) = \left[u,\ -\sqrt{3}\cdot u,\ 4\cdot u^{3}\right]$$

$$\mathrm{VECTOR}\left(\left[\left[u,\ \sqrt{3}\cdot u,\ 4\cdot u^{3}\right]\right],\ u,\ -1,\ 1,\ 0.01\right)$$

$$\mathrm{VECTOR}\left(\left[\left[u,\ -\sqrt{3}\cdot u,\ 4\cdot u^{3}\right]\right],\ u,\ -1,\ 1,\ 0.01\right)$$



We see two parabolic lines on the given surface and plot their respective maps on its dual (next page):

dualp(surf)

$$\left[-3\cdot u^2 - v^2,\ -2\cdot u\cdot v,\ -2\cdot u\cdot (u^2 + v^2)\right]$$

$$\left[-3\cdot u^2 - (u\cdot\sqrt{3})^2,\ -2\cdot u\cdot (u\cdot\sqrt{3}),\ -2\cdot u\cdot (u^2 + (u\cdot\sqrt{3})^2)\right]$$

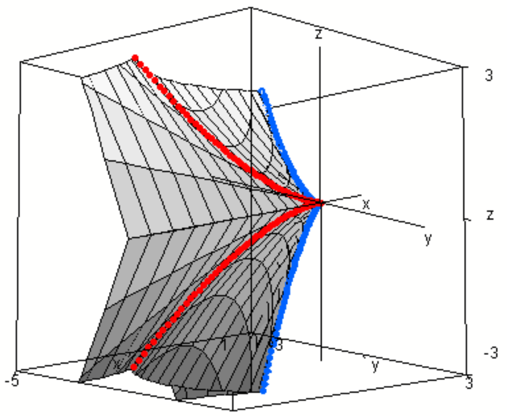
$$\left[-6\cdot u^2,\ -2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3\right]$$

$$\left[-3\cdot u^2 - (-u\cdot\sqrt{3})^2,\ -2\cdot u\cdot (-u\cdot\sqrt{3}),\ -2\cdot u\cdot (u^2 + (-u\cdot\sqrt{3})^2)\right]$$

$$\left[-6\cdot u^2,\ 2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3\right]$$

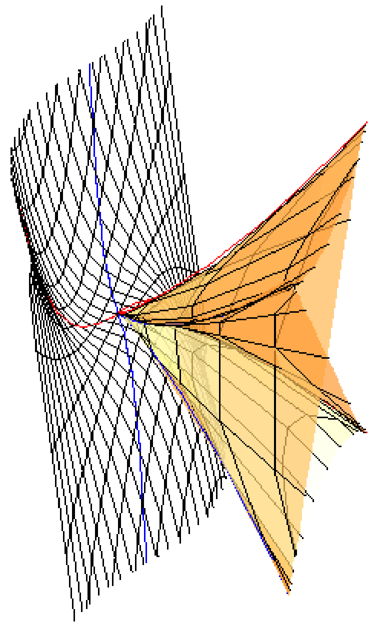
$$\text{VECTOR}\left(\left[\left[-6\cdot u^2,\ -2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3\right]\right],\ u,\ -1,\ 1,\ 0.01\right)$$

$$\text{VECTOR}\left(\left[\left[-6\cdot u^2,\ 2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3\right]\right],\ u,\ -1,\ 1,\ 0.01\right)$$



Same procedure with TI-NspireCAS:

$surf:=\{t,u,t^3+t\cdot u^2\}$	$\{t,u,t^3+t\cdot u^2\}$
$gauss\_curv(surf)$	$\dfrac{4\cdot (3\cdot t^2-u^2)}{(9\cdot t^4+10\cdot t^2\cdot u^2+u^4+1)^2}$
$surf\vert u=t\cdot\sqrt{3}$	$\{t,\sqrt{3}\cdot t,4\cdot t^3\}$
$surf\vert u=-t\cdot\sqrt{3}$	$\{t,-\sqrt{3}\cdot t,4\cdot t^3\}$
$dualsurf:=dualp(surf)$	$\{-3\cdot t^2-u^2,-2\cdot t\cdot u,-2\cdot t^3-2\cdot t\cdot u^2\}$
$dualsurf\vert u=t\cdot\sqrt{3}$	$\{-6\cdot t^2,-2\cdot\sqrt{3}\cdot t^2,-8\cdot t^3\}$
$dualsurf\vert u=-t\cdot\sqrt{3}$	$\{-6\cdot t^2,2\cdot\sqrt{3}\cdot t^2,-8\cdot t^3\}$



As expected, we receive two cuspidal edges. Unfortunately, we cannot plot the parabolic lines and their respective maps, the cuspidal edges as thick lines as we can do with DERIVE. I worked with parameters *t* and *u*, because these two parameters are requested for 3D-plots with TI-Npsire.

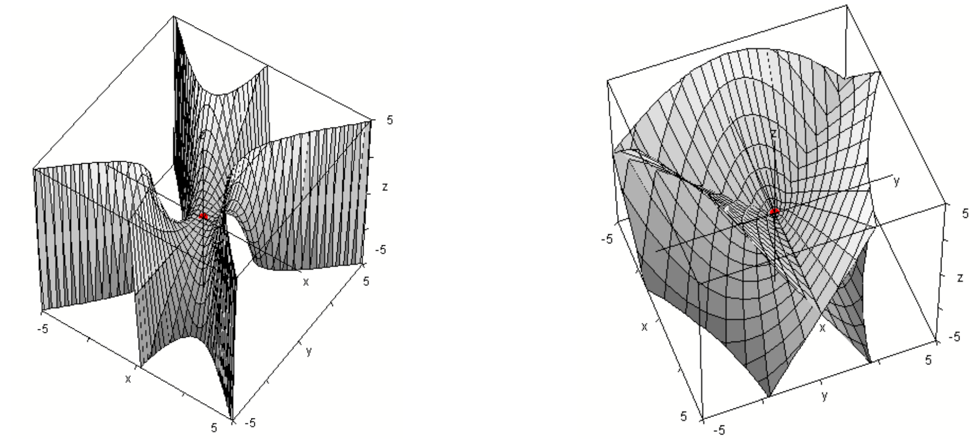
Morris writes:

If S has a cusp of Gauss at x, then the dual has a swallowtail point at x.*

Next surface has only one – isolated – parabolic point which results a *swallow tail point* on the dual:

$$f3 = x^3 - x\cdot y^2$$

$$\text{gauss\_curv}\left(\left[u,\ v,\ u^3 - u\cdot v^2\right]\right) = -\frac{4\cdot (3\cdot u^2 + v^2)}{(9\cdot u^4 - 2\cdot u^2\cdot v^2 + v^4 + 1)^2}$$



The swallow tail point (right)

A real swallow tail butterfly in our garden.

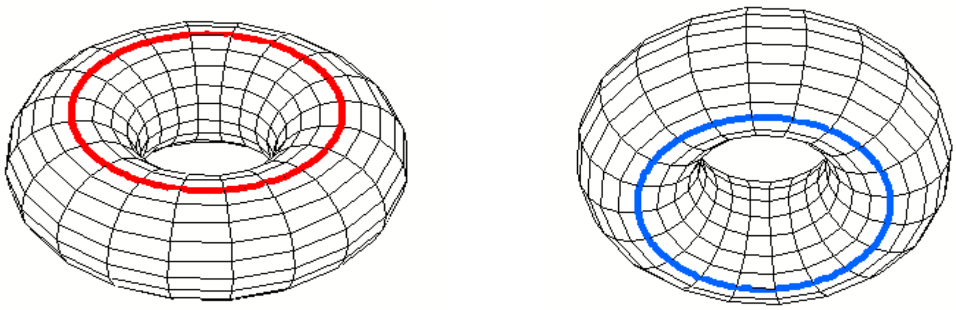
Once more the Torus – and its parabolic lines.

$$\text{gauss\_curv(torus)} = \frac{\text{COS(u)}}{\text{COS(u)} + 2}$$

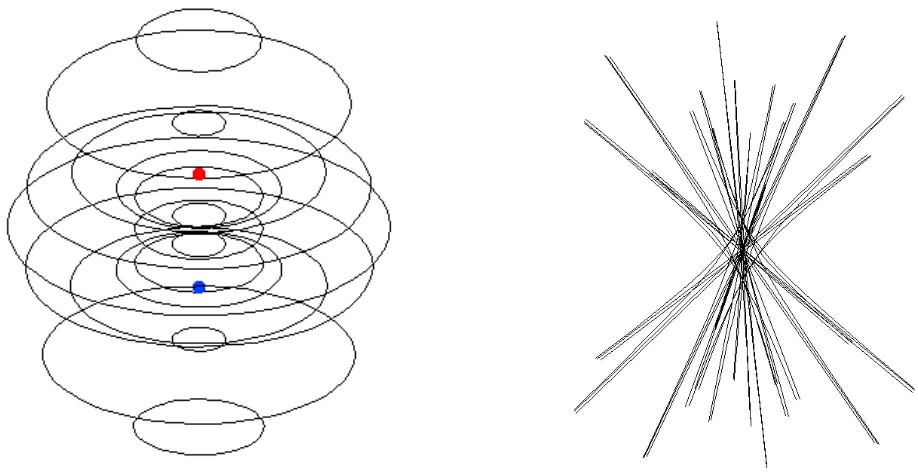
$$\text{SOLVE(COS(u) = 0, u)} = \left(u = \frac{3 \cdot \pi}{2} \lor u = -\frac{\pi}{2} \lor u = \frac{\pi}{2} \right)$$

$$\text{SUBST}\left(\text{torus, u, } \frac{\pi}{2} \right) = [2 \cdot \text{COS(v)}, 2 \cdot \text{SIN(v)}, 1]$$

$$\text{SUBST}\left(\text{torus, u, } \frac{3 \cdot \pi}{2} \right) = [2 \cdot \text{COS(v)}, 2 \cdot \text{SIN(v)}, -1]$$

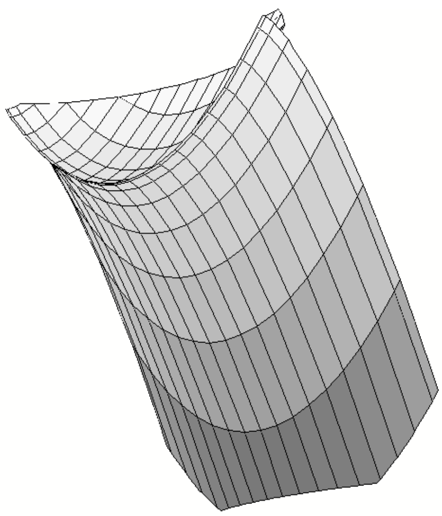


The parameter curves on the dual of the torus. The parabolic lines degenerate to one point each.



Finally, I had one question (for me, for students, for colleagues, …): How to find directly the cuspidal edge of a surface if there is one? Take $f(u,v) = (3u, 2v^2, -u^2 + 2v^3 + 2)$.

The answer is easy:
Cusps are there where the Gauss curvature is not defined. So set the denominator of the GC = 0.

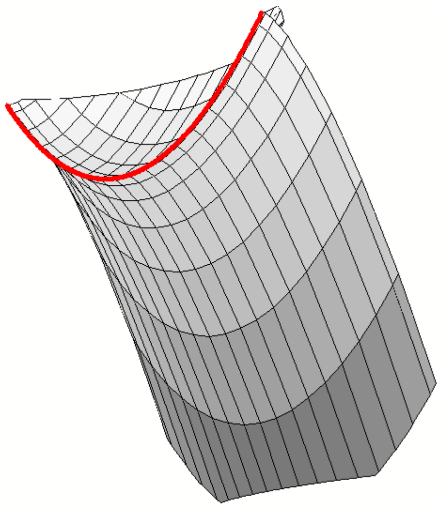


$$\text{gauss\_curv}([3\cdot u, -2\cdot v^2, -u^2+2\cdot v^3+2]) = -\frac{108}{v\cdot(16\cdot u^2+9\cdot(9\cdot v^2+4))^2}$$

$$\text{SOLUTIONS}(v\cdot(16\cdot u^2+9\cdot(9\cdot v^2+4))^2, v) = \left[0, \frac{2\cdot i\cdot\sqrt{(4\cdot u^2+9)}}{9}, -\frac{2\cdot i\cdot\sqrt{(4\cdot u^2+9)}}{9}\right]$$

$$\text{VECTOR}([[3\cdot u, 0, -u^2+2]], u, -5, 5, 0.01)$$

We have only one real solution $v = 0$.



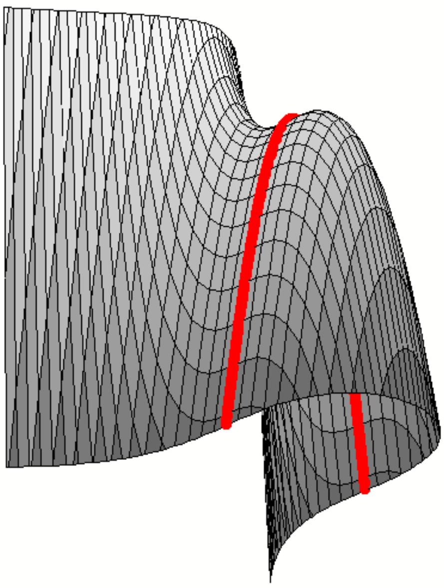
$$\text{dualp}([3\cdot u, -2\cdot v^2, u^2+2\cdot v^3+2])$$

$$\left[-\frac{2\cdot u}{3}, \frac{3\cdot v}{2}, -u^2-v^3+2\right]$$

$$\text{gauss\_curv}\left[-\frac{2\cdot u}{3}, \frac{3\cdot v}{2}, -u^2-v^3+2\right]$$

$$\frac{12\cdot v}{(9\cdot u^2+4\cdot v^4+1)^2}$$

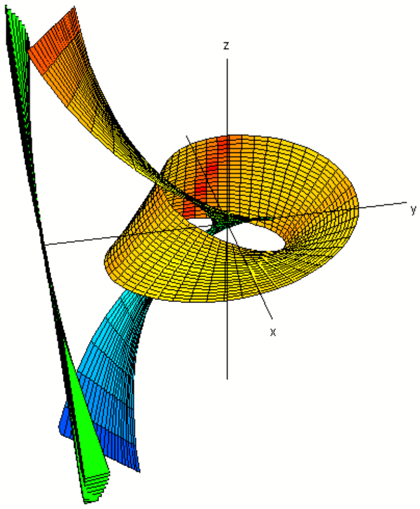
The dual has – as expected – for $v = 0$ a parabolic line.



I will finish with Dr Morris‘ favorite surface: the double twisted Moebius band and one possible form of its dual:

$$mb := \left[\cos(u)\cdot\left(1 + \frac{v}{2}\cdot\cos\left(\frac{u}{2}\right)\right),\ \sin(u)\cdot\left(1 + \frac{v}{2}\cdot\cos\left(\frac{u}{2}\right)\right),\ \frac{v}{2}\cdot\sin\left(\frac{u}{2}\right)\right]$$

dualp(mb)



My Conclusion:

The two short paragraphs in a journal from more than 20 years ago gave reason for an extended investigation. Every question answered offered new questions. I am quite sure that all of them can be treated in depth. I remained only on the surface. To cite David Halprin (Australia): “*I opened a can of worms*”.

I must admit that Morris’ extended paper – also from 2002 – opened further interesting insights for me. I started admiring fascinating pictures but then including the Gauss curvature it got more mathematical contents.

In my opinion, many parts of this talk could be treated in Upper Secondary classes. They might inspire students and Teachers for further investigations. Morris’ paper offers more ways to visualise surfaces and their duals which are well worth to be tried to realize with our tools.

It was funny for me that I stepwise remembered many facts which I have heard long ago. Computer Algebra and graphing tools helped immensely.

References:

www.singsurf.org/papers/dual/dual.pdf (can be obtained from the author of this paper)

Alfred Gray, Modern Differential Geometry

R. Haas & J. Böhm, Cello Tangents, DERIVE Newsletter 125

DP Graph, http://www.dpgraph.com/

https://github.com/RichardMorris/SingSurf

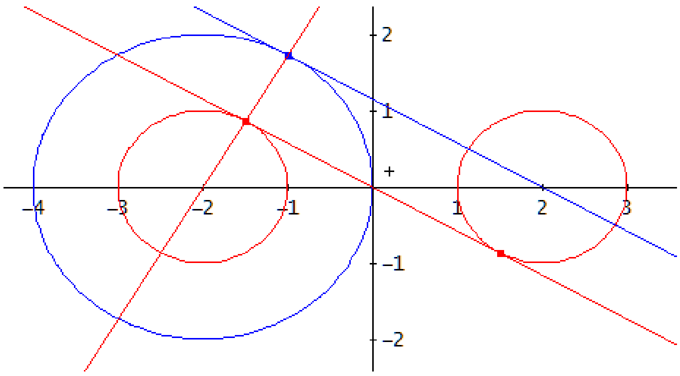
Roman Hašek, Exploration of dual curves using a dynamic geometry, and computer algebra system, ACA 2018

Appendix

Remember the double points of the dual curves of the quartic and quantic above. What about self-intersection of dual surfaces?

Dr. Morris writes in his extended paper: *"If there is a plane tangent to S at x and y then S* has a self-intersection at x* = y*."*

As it is not so easy to calculate bi-tangent planes, I took again the torus for demonstrating this property of dual surfaces. It gives another opportunity for a students' project.



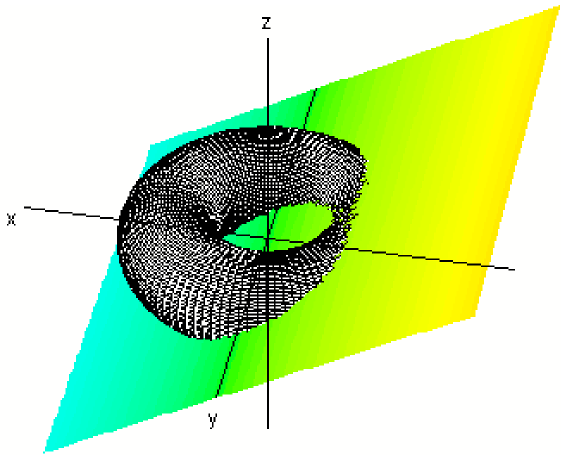
We find the double tangent of the cross section of the torus following the well-known construction of the common tangent on two circles.

$$\left[-\frac{3}{2}, \frac{\sqrt{3}}{2}\right]$$

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\cdot\left(x + \frac{3}{2}\right)$$

$$pl10 := \left[-\frac{3}{2}, 0, \frac{\sqrt{3}}{2}\right] + u\cdot[0, 1, 0] + v\cdot\left[1, 0, -\frac{1}{\sqrt{3}}\right]$$

Transferring the result in 3D space gives one bi-tangent plane pl10.



Back in the 3D-world, we have to find out the respective u and v values for the point - in order to apply my functions -, which are $u = 2\pi/3$ and $v = \pi$.

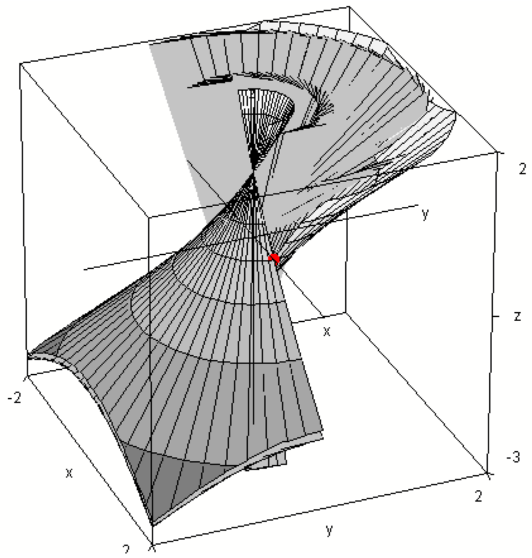
Then we calculate the tangent plane in this point and its corresponding point on the dual surface, which should turn out to be a point of the self-intersecting curve!

$$-\frac{3\cdot x}{4} - \frac{3\cdot\sqrt{3}\cdot z}{4} = 0$$

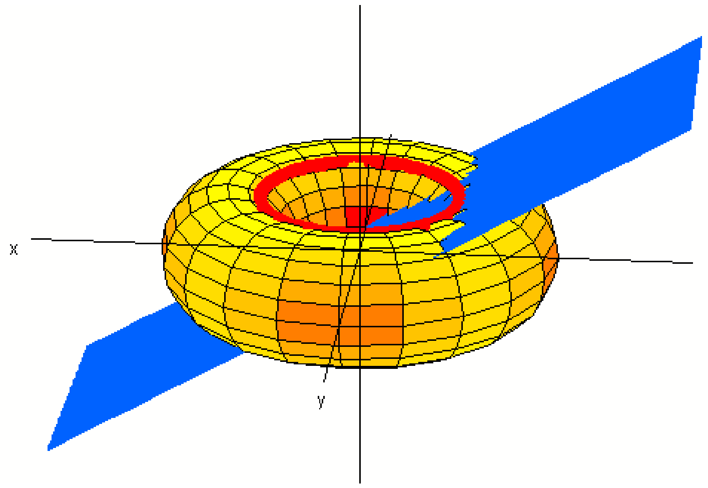
$$-\frac{1}{\sqrt{3}}\cdot x$$

$$\mathrm{dualf}\left(-\frac{1}{\sqrt{3}}\cdot x\right)$$

$$\left[\frac{\sqrt{3}}{3},\ 0,\ 0\right]$$



The plot shows only a part of the dual surface to make this point (red) better visible.



The torus together with the bi-tangent plane and the set of all points on the torus with points giving double tangential planes. The mapping of the red circle should result in the self-intersection of the dual surface.

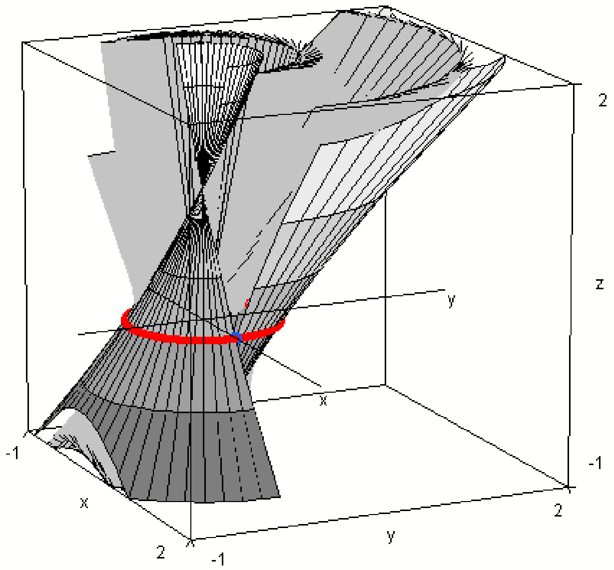
$$\left[\frac{\mathrm{COS}(u)\cdot\mathrm{COS}(v)}{\mathrm{SIN}(u)},\ \frac{\mathrm{COS}(u)\cdot\mathrm{SIN}(v)}{\mathrm{SIN}(u)},\ \frac{\mathrm{COS}(u)^3 + 2\cdot\mathrm{COS}(u)^2 + \mathrm{COS}(u)\cdot(\mathrm{SIN}(u)^2 + 4) + 2}{\mathrm{SIN}(u)\cdot(\mathrm{COS}(u) + 2)}\right]$$

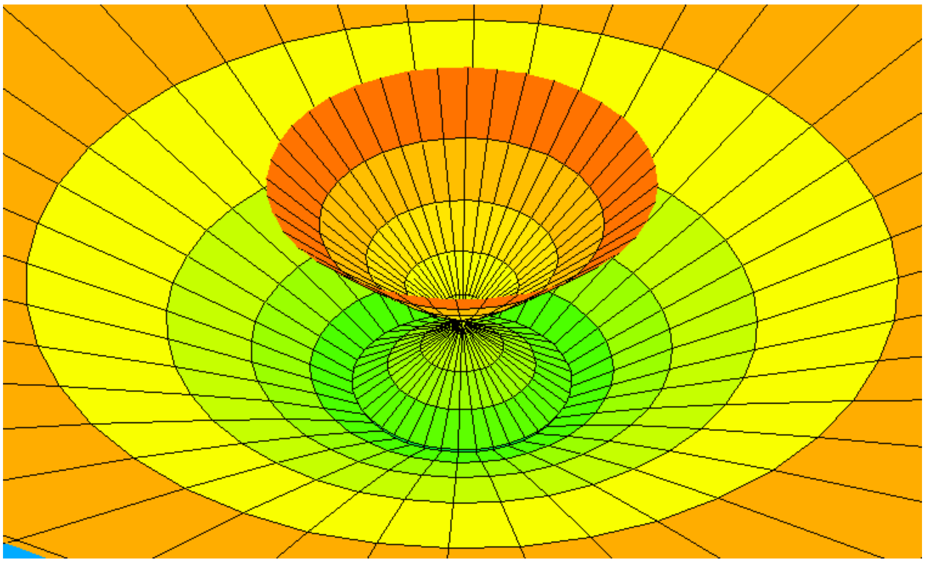
$$\left[\frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\mathrm{COS}(v)}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)},\ \frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\mathrm{SIN}(v)}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)},\ \frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)^3 + 2\cdot\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)^2 + \mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\left(\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)^2 + 4\right) + 2}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)\cdot\left(\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right) + 2\right)}\right]$$

$$\left[-\frac{\sqrt{3}\cdot\mathrm{COS}(v)}{3},\ -\frac{\sqrt{3}\cdot\mathrm{SIN}(v)}{3},\ 0\right]$$

$$\mathrm{VECTOR}\left(\left[\left[-\frac{\sqrt{3}\cdot\mathrm{COS}(v)}{3},\ -\frac{\sqrt{3}\cdot\mathrm{SIN}(v)}{3},\ 0\right]\right],\ v,\ 0,\ 2\cdot\pi,\ 0.01\right)$$

Parameter curve with $u = 2\pi/3$:





DPGraph-view from above into the interior of the dual. We can see the self-intersecting curve (horizontal circle).