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Assessment of students' knowledge and abilities in undergraduate mathematics

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In memory of Eugenio Roanes-Lozano

In the triad Teaching-Learning-Assessment (TLA) the components are to be considered in tandem, not standalone. The activities are to be interrelated, because what gets assessed is what gets taught. That is why we need to focus on: purposeful Teaching, purposeful Learning and purposeful Assessment. A TLA process could help the students build habits and qualities of mind that are useful for the real life and, above all, for their work.

Typically, teachers ask the students increasingly challenging questions to test their comprehension of a given material. In the textbooks, the most common verbs are find, determine, calculate, solve, explore. These abilities show that the student has accomplished the goal "Remember, Understand and Apply" (in Blooms taxonomy), i.e. Lower Order Learning (LOL); the latter are really necessary and important and have to be assessed. However in technology enriched teaching-learning environment there are opportunities to interpret this goal and change the way of its accomplishment. Higher Order Learning (HOL: development of abilities to analyze, synthesize, create) is to be also achieved and assessed. Care must be taken to changes: the things we think are changing aren't always what's changing.

The students have to learn certain things, to achieve learning outcomes, as well as to learn how to learn. New professions require not only knowledge and skills, but also logical, critical and creative thinking. The student has to build up as well the habit to do things in sequence in order to develop an organized and disciplined mind. The assessment of students' knowledge and abilities has to assure the accomplishment of these qualities. Here we share thoughts and experience in this direction.

Using a CAS any Taylor polynomial of interest can be obtained. Then instead of its determination further questions for testing the deepness of students' knowledge of this concept can be set up. For instance, consider the following tasks.

Task 1. Given the 5th degree Taylor polynomial $T_5(x) = \dfrac{x}{2} + \dfrac{x^2}{8} + \dfrac{x^3}{24} + \dfrac{x^4}{64} + \dfrac{x^5}{160}$ at the point $x_0 = 0$ of the function $f(x) = \ln \dfrac{2}{2-x}$.

(a) Show that the first term is correct.

(b) Calculate an approximation of $f(0.5)$ (or of $\ln \dfrac{4}{3}$) using the second degree Taylor polynomial $T_2(x)$ at the point $x_0 = 0$.

(c) Evaluate $f^{(5)}(0)$.

Task 2. The function $f(x) : R_+ \to R_+$ satisfies the following conditions:

$$f(4) = 2, f'(4) = 0.25, f''(x) = -\frac{0.25}{x\sqrt{x}}.$$

(a) Find the polynomial of Taylor of 3rd degree $T_3(x)$ of $f(x)$ at the point $x_0 = 4$.

(b) Calculate an approximation of $f(4.2)$ using the second degree Taylor polynomial $T_2(x)$ at the point $x_0 = 4$.

(c) Determine the function $f(x)$.

The aim of such kind of questions is to help students consolidate key knowledge about Taylor polynomials: their construction, their applications: for calculating values of functions, for solving approximately integrals and differential equations. The solution also aims at HOL and development of the habit to solve problems not just any how, to work (consequently, perform any activity) smarter not harder. In the presentation the approach to the solution of the above tasks will be considered.

In case of Fourier series similar questions can be formulated. When the student is checking up, for instance, the correctness of a term he/she develops the habit to control the results using different prototypes (analytical, numerical, graphical) including those obtained by application of software.

Task 3. Given the Fourier series $f(x) = \dfrac{\pi}{2} + \dfrac{4}{\pi}\cos x + \cdots + a_n \cos(nx) + \ldots$ for the periodic function $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$, $f(x + 2\pi) = f(x), \; x \in R.$

Show that the second term is correct. Sketch the graph of f and justify the form of the series.

CASs allow to formulate questions based on visual information.

Task 4. The area D is bounded by the curves $y = 9 + 3x$ and $y = 9 - x^2$ (Fig. 1). Describe D in terms of double inequalities in two ways. Calculate the integral $\iint\limits_D dydx$ and interpret the result.

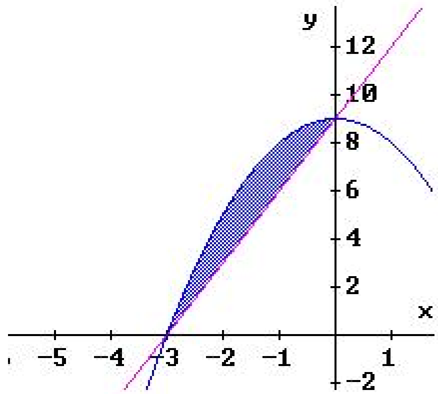

Fig. 1

"Work smarter, not harder" is related to effective solutions. Asking right questions ensures effective learning. In connection to effectiveness the following cases will be considered in the presentation.

1) Evaluate the second order partial mixed derivative of the function

$f(x, y) = \ln(2x + 5) - 2x \arctan(2x) + e^{5y} + \sin(xy) + 24$ at the point $P\left(\frac{\pi}{2}, 1\right)$.

2) Solve: $\displaystyle\int \frac{\mathrm{d}x}{x\ln(x)}$; $\displaystyle\int x\sqrt{16 - x^2}\mathrm{d}x$; $\displaystyle\int_0^1 \sqrt{1 - x^2}\mathrm{d}x - \int_0^{0.5} \sqrt{\frac{1}{4} - x^2}\mathrm{d}x$;

$$\int_{-0.5}^{0.5} (12x^3 + x\cos(x) - 6\pi\cos(3\pi x))\mathrm{d}x.$$

By the choice of an approach to the solution the basic question is whether to apply a method if the solution can be obtained by direct application of definitions, properties or/and graphical images.

Mathematics and computer science education is of great importance to society. Any activity to stimulate and provoke the interest of young people to these fields will be for the benefit of society. This would produce added value to this education. The second author of this paper proposed the idea of organizing national student contest in mathematics with application of CASs. He shared it with university teachers keen to implement CASs in the teaching-learning process. As a result of teachers' and students' enthusiasm the experimental competition in Computer Mathematics took place in 2011 in Bulgaria [1]. The 9th edition of CompMath was held in October 2022.

It has to be mentioned that in the past several years Geo-Gebra is intensively used at Bulgarian math high schools. Advanced students and students who prefer doing mathematics in technology-supported environment are the majority of participants in this competition. All they need to be stimulated by additional activities at the universities: establishment of initiatives such as informal education, visiting lecturers - distinguished professionals in computer mathematics and computer science, clubs for exchange of ideas and experience. Based on their achievements they need to be advised to acquire additional learning material, so that to develop their full potential.

CompMath is now a traditional annual forum for students at Bulgarian universities. It proved to be useful for stimulating students' interest in mathematics and the opportunities of CASs for solving both theoretical and applied problems. It helps to create new as well as best practices in mathematics education: they could serve as models for suitable purposeful problems and assessment criteria.

The participants in the CompMath are given 30 mathematical problems to be solved within four hours. Based on their bachelor degree program they are divided into two groups:

- Group A: Mathematics, Informatics, Computer Science;

- Group B: Engineering, Natural Sciences.

All topics from the mathematics courses are covered. The students solve the problems in different ways depending on the level of their mathematical knowledge and programming skills. In the presentation some interesting problems and solutions created by participants will be demonstrated [2]. Mathematica, Maple, Maxima, Derive and MATLAB are mostly used in CompMath.

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undergraduate mathematics, CASs, assessment, competition in computer mathematics

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