Undecidability of Noncommutative Ideal Membership and Counterexamples of Operator Statements

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Computations with identities of linear operators can be translated into symbolic computations with noncommutative polynomials in free algebras. Through this translation, proving the correctness of operator identities reduces to verifying ideal membership of such polynomials [4] and [5]. While verifying ideal membership in free algebras is always possible using noncommutative Gröbner bases, disproving it is in general undecidable [6]. Nevertheless, in practice, one can often refute ideal membership by constructing explicit counterexamples (in the form of matrices).

In this talk, we first outline the undecidability of the ideal membership problem in free algebras. While one would think that ideals with undecidable membership problem are monstrous, complicated objects, already Tseitin [2] and [7] provided a simple example of such an ideal, which we discuss in the talk. We also present a method to compute explicit matrix counterexamples by combining SAT solving and algebraic techniques (Hensel lifting and rational reconstruction). As a special case, we discuss how to compute simple counterexamples containing only 0 and ± 1 as entries. These methods are implemented in SageMath as part of the operator_gb package [1]. We illustrate them on examples coming from the theory of generalized inverses [3].

References

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