Separated Variables on Plane Algebraic Curves

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We consider equations of the form

$$r(x, y) + q(x, y)p(x, y) = f(x) - g(y),$$

for rational functions $r(x, y)$, $q(x, y)$, $p(x, y)$, $f(x)$ and $g(y)$ in x and y over $\mathbb{K}$, and explain how they can be solved based on the ideas developed in [1] to [3]. The procedure we present reduces the non-linear problem to a linear one. However, the procedure is just a semi-algorithm. It terminates, whenever the equation has a non-trivial solution, but it may not, if there is none. Termination depends on a dynamical system on the curve associated with p and the location of the poles of r thereon. It is still an open question how the semi-algorithm could be turned into an algorithm.

The problem has a field theoretic interpretation. Let $\mathbb{K}(x, y)$ be the field generated by elements x and y satisfying the (only) relation $p(x, y) = 0$, and let $\mathbb{K}(x)$ and $\mathbb{K}(y)$ be the subfields generated by x and y, respectively. Then the above equation has a (non-trivial) solution if and only if $r(x, y)$ is an element of $\mathbb{K}(x) + \mathbb{K}(y)$. There are two particular cases that are interesting in themselves: the case $r = 0$, and the case $g = 0$. The former corresponds to the problem of computing the intersection of $\mathbb{K}(x)$ and $\mathbb{K}(y)$, the latter to the problem of deciding whether $r(x, y)$ is an element of $\mathbb{K}(x)$ and finding all representations thereof in terms of x.

The problem arises in enumerative combinatorics, when solving discrete differential equations by reducing partial DDEs to systems of ordinary ODDEs [4]. It also arises in parameter-identification problems in ODE models [5], and in problems of image recognition [6].

References

[1] Manfred Buchacher, Manuel Kauers, and Gleb Pogudin. Separating Variables in Bivariate Polynomial Ideals. *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, pages 54-61, 2020.

[2] Manfred Buchacher. Separating Variables in Bivariate Polynomial Ideals: the Local Case. *arXiv preprint*, arXiv:2404.10377, 2024.

[3] Manfred Buchacher. Separated Variables on Plane Algebraic Curves. *arXiv preprint*, arXiv:2411.08584, 2024.

[4] Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel. Counting quadrant walks via Tutte's invariant methods. *Discrete Mathematics & Theoretical Computer Science*, 2020.

[5] Alexey Ovchinnikov, Anand Pillay, Gleb Pogudin, and Thomas Scanlon. Computing all identifiable functions of parameters for ODE models. *Systems & Control Letters*, 2021.

[6] Anna Katherina Binder. Algorithms for Fields and an Application to a Problem in Computer Vision. *PhD Thesis.* Technische Universität München, 2009.