Recent Advancements in Noncommutative Gröbner Basis Software

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In recent years, noncommutative Gröbner bases in free algebras (also known as Gröbner-Shirshov bases) have found important applications in areas such as (linear) control theory [9], automated theorem proving for operator statements [3], [4], [10] and [12], as well as in graph [11] and game theory [8].

These applications crucially rely on the ability to compute Gröbner bases in free algebras efficiently. While software for commutative Gröbner basis computations has seen remarkable progress in recent years (see [1] and references therein), noncommutative tools seem to lag behind. They often lack the same level of efficiency and sophistication, and mostly rely on outdated algorithms and data structures.

In this talk, we give an overview of existing software for Gröbner basis computations in free algebras. We also present `f4ncgb` [5], a new open-source C++ library for this task. Moreover, we discuss recent algorithmic improvements that could be integrated into existing tools in the future, in particular, signature-based algorithms [2] and [6] and support for more general coefficient domains such as the integers [7].

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