Homotopy Methods for Computing Roots of Mandelbrot Polynomials

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The Mandelbrot polynomials are recursively defined as:

$$p_0(z) = 1, \quad p_{n+1}(z) = z p_n(z)^2 + 1,$$

and serve as a test problem for exploring the computation of roots in highly structured, recursively defined polynomials. The roots of these polynomials can be computed by constructing a companion matrix, referred to as the Mandelbrot matrix, whose eigenvalues correspond to the roots of the polynomial. The Mandelbrot matrices are recursively defined as:

$$\mathbf{M_{n+1}} = \begin{bmatrix} \mathbf{M_n} & \mathbf{0} & -\mathbf{c}_n \mathbf{r}_n \\ \mathbf{r}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_n & \mathbf{M_n} \end{bmatrix},$$

where $\mathbf{M_1} = [-1]$. The size of $\mathbf{M_n}$ is $d = 2^{n-1}$, and $\mathbf{c}_n$ and $\mathbf{r}_n$ are vectors of size d, given by:

$$\mathbf{c}_n = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & -1 \end{bmatrix}.$$

This talk investigates the use of homotopy methods as a viable approach to compute the eigenvalues of Mandelbrot matrices by exploiting their recursive structure. Homotopy techniques offer a divide-and-conquer framework that deforms a simpler base matrix into the target matrix, enabling symbolic or numerical tracking of eigenvalues across iterations. Additionally, we extend this approach to Mandelbrot-like matrices, such as Fibonacci-Mandelbrot, Narayana-Mandelbrot, and Euclid matrices, which share similar recursive properties. The results highlight the potential of homotopy methods to efficiently solve eigenvalue problems in structured and recursively defined matrices.