A purity theorem for Mahler equations

Julien Roques
Université Lyon 1, France

30th Applications of Computer Algebra - ACA 2025

Let $p \geq 2$ be an integer. This talk concerns (linear) p-Mahler equations, *i.e.*, linear functional equations of the form

$$a_0(z)f(z) + a_1(z)f(z^p) + \cdots + a_d(z)f(z^{p^d}) = 0, \tag{1}$$

where the coefficients $a_0, \ldots, a_d$ belong to $\overline{\mathbb{Q}}(z)$ and satisfy $a_0 a_d \neq 0$. For instance, the generating series of p-automatic sequences—or more generally, of p-regular sequences—satisfy such equations.

Hahn series play a key role in the study of p-Mahler equations. Roughly speaking, Hahn series generalize Puiseux series by allowing arbitrary rational exponents of the indeterminate, provided that the set that supports them is well-ordered. Their significance in our context is made clear by the following result: the difference field $(\mathscr{H}, \varphi_p)$, where $\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$ is the field of Hahn series with coefficients in $\overline{\mathbb{Q}}$ and value group $\mathbb{Q}$ and where φ_p is the field automorphism of $\mathscr{H}$ sending $f(z)$ on $f(z^p)$, has a difference ring extension $(\mathcal{R}, \varphi_p)$ with field of constants $\mathcal{R}^{\varphi_p} = \{f \in \mathcal{R} \mid \varphi_p(f) = f\}$ equal to $\overline{\mathbb{Q}}$ such that

- for any $c \in \overline{\mathbb{Q}}^{\times}$, there exists $e_c \in \mathcal{R}^{\times}$ satisfying $\varphi_p(e_c) = ce_c$;
- there exists $\ell \in \mathcal{R}$ satisfying $\varphi_p(\ell) = \ell + 1$;
- any p-Mahler equation of the form (1) has d solutions $y_1, \ldots, y_d \in \mathcal{R}$ that are $\overline{\mathbb{Q}}$-linearly independent and of the form
$$y_i = \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geq 0}} f_{i,c,j} e_c \ell^j, \tag{2}$$
where the sum has finite support and the $f_{i,c,j} \in \mathscr{H}$ satisfy p-Mahler equations.

In this talk, we will focus on the growth of the logarithmic Weil height of the coefficients of the Hahn series that arise when solving p-Mahler equations. We will report on recent joint work with C. Faverjon, in which:

- we show that five distinct asymptotic growth behaviors can occur, thereby generalizing a previous result by B. Adamczewski, J. P. Bell, and D. Smertnig about p-Mahler series;
- we establish a purity theorem reminiscent of classical purity theorems for G-functions due to D. and G. Chudnovsky, and for E-functions (and more generally, for holonomic arithmetic Gevrey series) due to Y. André.