# Envelopes of Circles Centered on a Cubic-Quadratic: A Dynamic Exploration of their Topology and Singular Locus

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## Abstract

This study investigates the envelopes of a 1-parameter family of circles with constant radius $r$ (for various values of $r$), centered on the cubic-quadratic curves $\mathcal{Q}_a$, defined by $y^2 = x^3 + ax$, where $a \in \mathbb{R}$. We analyze how the topology of the envelope changes as a function of $a$ and $r$. Networking Dynamic Geometry Systems (DGS) with Computer Algebra Systems (CAS) and Generative AI, we identify complex singular structures (cusps, crunodes, and self-tangencies). This work enhances the necessity to develop automatic communication between between DGS and CAS to validate conjectures with rigorous algebraic proof [7].

## 1 Theoretical Background

An envelope of a family of plane curves $\mathcal{P}_s$ can be defined impredicatively as a curve $\mathcal{E}$ tangent to a unique curve of the family at every point, the points of contact being different for different values of $s$. Analytically, for a family given by an equation $F(x, y, s) = 0$, the envelope is the solution set of the system of equations $F(x, y, z) = \frac{\partial}{\partial s} F(x, y, s) = 0$.

For a regular curve $\gamma(s)$ parametrized by arc length, the envelope $\gamma_r$ of circles with fixed radius $r$ is the union of two components:

$$\gamma_r^{\pm}(s) = \gamma(s) \pm rN(s) \tag{1}$$

where $N(s)$ is the unit normal. However, if the base curve $\gamma$ possesses a singular point (such as a cusp), the tangent vector vanishes, potentially causing the entire circle to enter the envelope.

## 2 The Networking of Technologies

The complexity of high-degree algebraic curves (reaching degree 14 in this study) requires a multi-tool approach.

- **GeoGebra-Discovery:** Provides automated commands like `Envelope` and `Locus`. It allows for real-time dragging of centers to observe topological shifts visually; see [2].
- **Computer Algebra Systems (Maple/Mathematica):** Used for Gröbner bases computations and elimination. CAS is essential when DGS works numerically.
- **Generative AI (ChatGPT 5.2):** Acts as a "tutor" or assistant to generate code and factorizations. We check its reliability, noting that AI-generated outputs must always be verified by a CAS. We experience advances with new versions, with regards to [1, 3].

Networking between different kinds of software provides a rich environment for exploration-conjecture-validation [4, 6], as it draws on the strengths of each package. Of course, the community wishes to have benefit of direct communication between the kinds of software [7].

# 3 Topological Analysis by Parameter $a$

The curve $\mathcal{Q}_a : y^2 = x^3 + ax$ exhibits three distinct behaviors, according to the values of $a$.

## 3.1 The Cuspidal Case ($a = 0$)

The curve $\mathcal{Q}_0$ has a cusp at the origin. For $r = 1$, the envelope is an algebraic curve of degree 10, namely the union of a unit circle $x^2 + y^2 - 1 = 0$ and an octic variety. Exploration with software reveals that only half of this circle is relevant to the impredicative envelope. See Figure 1.

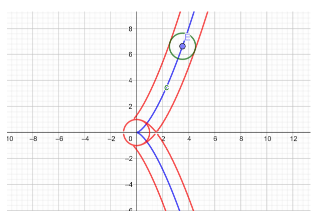

Figure 1: a=0,r=1

## 3.2 The Smooth Single-Component Case ($a > 0$)

For $a = 1$, the curve $Q_a$ is smooth. It appears that the envelope (an algebraic curve of degree 14) is not necessarily smooth. Standard plotting often shows discontinuities near $(1, 0)$, which, upon high-level magnification, reveal a "curved triangle" bounded by two cusps and a crunode (self-intersection). For general $a > 0$ we have the following result, which gives a linear connection between $a$ and $r$ with respect to the domain of smoothness of the envelope.

**Theorem 1.** *Let $a > 0$. The envelope of $\gamma$ is smooth if and only if $r < \frac{a}{2}$*

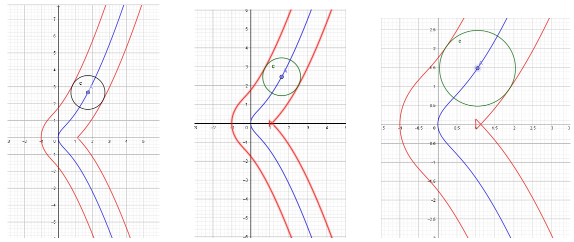

Figure 2: Three screenshots of the same envelope: a curved triangle appears

### 3.3 The Two-Component Case ($a < 0$)

For $a < 0$, the curve $Q_a$ is the union of an open component and a disjoint loop. This leads to the most complex topologies, where different envelope components intersect.For $a = -1$, we observe internal components with self-tangency points and multiple curved triangles.

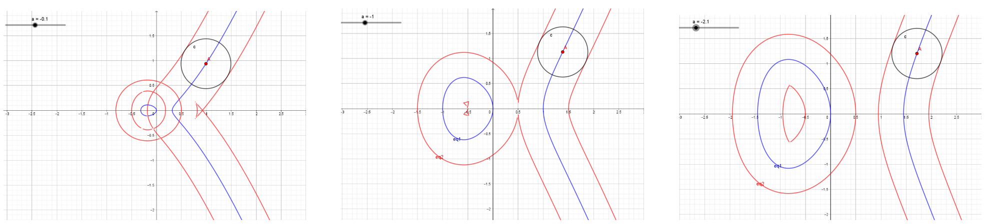

Figure 3: Exploration of envelopes for $r = 1$

### 3.4 Remarks

Plotting tools often leave "blank zones" in plots due to their difficulties near singularities. Here and in future work, we focus on using successive differentiation and curvature-based algorithms in Maple to provide a complete taxonomy of these singularities. This study reinforces the necessity of networking between different kinds of software to enjoy the strongest abilities of each one, enhancing also the importance of man-and-machine and of machine-and-machine communication [5, 7]. For example, GeoGebra's **Locus** command is numerical and **LocusEquation** is symbolic (we used them for the figures above); this means that the 2nd one may not be defined in certain situations where the 1st one gives an answer. In such a case, we may obtain a graphical answer but no equation. Another CAS is then needed for symbolic computations. Anyways, going back to the DGS may be important to explore relevance of components (the symbolic answers gives an algebraic variety defined as a closure in Zariski topology) .

The family of curves under study is given a priori by a cartesian equation. For numerous computations, it is worth to have a parametric presentation. For $Q_0$, such a parametrization is easy to derive, as the curve has a single singular point. For other values, the task is more complicated. Generally, we use the CAS-based methods described in [8].

## References


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