Be careful when adding an integration constant

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Key words: *Indefinite and definite integrals; piecewise defined functions, continuous antiderivatives*

How many times have students in a calculus course been reminded to add a constant of integration at the end of an indefinite integral? Many times, no doubt. All the while, they often forget that the addition of this crucial constant of integration stems from the theorem stating that two antiderivatives of the same **continuous** function over an **interval** differ by a constant. And all the while, they use integral tables that violate the aforementioned theorem and disregard the contributions of computer algebra systems from the last three decades!

They were told not to forget to add a constant of integration at the end of the calculation of an indefinite integral, but they rarely had the opportunity to calculate the constant of integration required to find a *continous* antiderivative of a piecewise continuous function. Or even to find the constant by which two antiderivatives of the same continuous function differ on a given interval.

Of course we must not lose sight of the following fact, as the authors Corless, Jeffrey and Stoutemyer have rightly noted in [12]:

> "*Asking students, on their first encounter with antiderivatives, to worry about the continuity of their answer, in addition to other worries, might seem unreasonable. Asking computer algebra systems to cater to the needs of first year students as well as the needs of people who solve differential equations with parameters might also seem unreasonable, without some sort of switch to "expert mode".*

From a personal point of view, the development of the Rubi system by the late Albert Rich and, well before that, the legendary computer algebra system ***Derive***, motivated me to write and talk about the stubbornness of the authors of integral calculus manuals regarding the integral tables they use. Many talks about this were given at recent ACA conferences. Also, a daily use of Texas Instruments CAS handheld and software is a good opportunity to continue to try to convince colleagues. It seems I did not succeed. Why not try again? But it won't be easy. Some 30 years, a paper ([7]) published by mathematicians involved with ***Derive*** and *Maple* was concluded with the following message:

> "*The correct integration of piecewise-continuous functions is not solely a CAS issue. The average user of a CAS has received little instruction from elementary mathematics books on working with functions as simple as $|x|$ – indeed no table of integrals contains an entry for this functions – and without that background users might be slow to accept them. In addition, the integration of piecewise functions requires users to understand the difference between integration with respect to a complex variable and with respect to a real variable. There has already been a significant impact by CAS on the practice and teaching of mathematics, and piecewise continuous functions could be another area in which CAS will lead the way".*

We will attempt to show that if calculus textbooks include more examples of piecewise defined functions – these are so important in applied mathematics and in engineering –, the understanding of adding the constant of integration would likely be improved. The examples that will be presented

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were motivated by the following observation: several students taking a course in differential equations have difficulty finding the required constant of integration in a rectilinear motion problem. Especially if the given velocity of the object moving in a straight line is defined by a piecewise continuous function.

So we will let $v(t)$ be the velocity of an object moving horizontally along the x-axis with $a \leq t \leq b$. The position of the object at time t will be $x(t)$and x_0 will be the initial position: $x(a) = x_0$. The change in position (displacement) between a and b is given by

$$\int_a^b v(t)dt \tag{1}$$

The examples will consist in finding the displacement and the position of the object. Some velocity functions will be everywhere continuous, some will be piecewise continuous. We will show the problems students are facing when they want to find the displacement and are using the fundamental theorem of calculus ("FTC": the first or the second form). And why, using a CAS (handheld or on a computer), can help students. Let us recall this important theorem by formulating it using the benefits of computer algebra from the last 30 years:

Theorem: *Fundamental theorem of calculus* (first form). Let f be a continuous function defined over the interval $[a,b]$. Let F be continous on $[a,b]$, differentiable on $]a,b[$ and such that $F'(x) = f(x)$ for $a < x < b$. Then [1]

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \tag{2}$$

It is worth reminding students that a *continuous* antiderivative F of f must be chosen in (2).

Theorem: *Fundamental theorem of calculus: construction of an indefinite integral* (second form). Let f be a bounded function defined over the interval $[a,b]$ and Riemann integrable [2] on the interval $[a,b]$. Let g be defined by

$$g(x) = \int_a^x f(t)dt \quad (a \leq x \leq b). \tag{3}$$

Then $g'(x) = f(x)$ at every point of continuity of f.

- ☞ In particular, if f is everywhere continuous over $[a,b]$, then g is also continuous over $[a,b]$ and differentiable over $]a,b[$ with $g'(x) = f(x)$ for $a < x < b$.
- ☞ And if f is only piecewise continuous over $[a,b]$, the function g will also be continuous over $[a,b]$ and diffentiable except at the points of discontinuity of f.

We rarely see, in calculus textbooks, examples of using the fundamental theorem of calculus (2) with a piecewise continuous function such as the signum function or the Heaviside (step) function. It is an excellent opportunity to reinforce the understanding of the FTC. In a differential equation course, the first type of differential equations students are facing is the so-called directly integrable differential equations, that is those having the form $\frac{dy}{dt} = f(t), y(t_0) = y_0$. This is a nice opportunity to recall some integration techniques from calculus and, in the case it would be too long to integrate by hand, how to use a CAS (let us recall that every student in my institution has a TI-Nspire CX II CAS handheld on his desk).

The first form of the fundamental theorem of calculus will yield the displacement between a and b: if $x(t)$ is a continuous antiderivative of $v(t)$ – that is, $x(t)$ is continuous and $\dfrac{d}{dt}x(t) = v(t)$ at each point where $v(t)$ is continuous –, then

$$\int_a^b v(t)dt = x(b) - x(a) \tag{4}$$

[1] $F(a)$ and $F(b)$ are usually computed by right-hand and left-hand side limits respectively.
[2] My own students are told this means f is continuous *almost everywhere* on $[a,b]$.

The second form of the fundamental theorem of calculus yields a nice formula for the position $x(t)$ of the object at any time t for $a \le t \le b$:

$$x(t) = x_0 + \int_a^t v(\tau)\,d\tau \quad (a \le t \le b) \tag{5}$$

In the case where the velocity function is a piecewise continuous (see Figure 1), students using a CAS (handheld or on a computer) have a nice opportunity to (try to) understand the importance of finding a continuous antiderivative for the velocity. The presentation will show how computer algebra systems like TI-Nspire CAS and Maple can improve the understanding of different concepts. And another advantage of using a CAS is the possibility to use animation: the "second form" of the FTC benefits from this as Figure 2 will show during the talk.

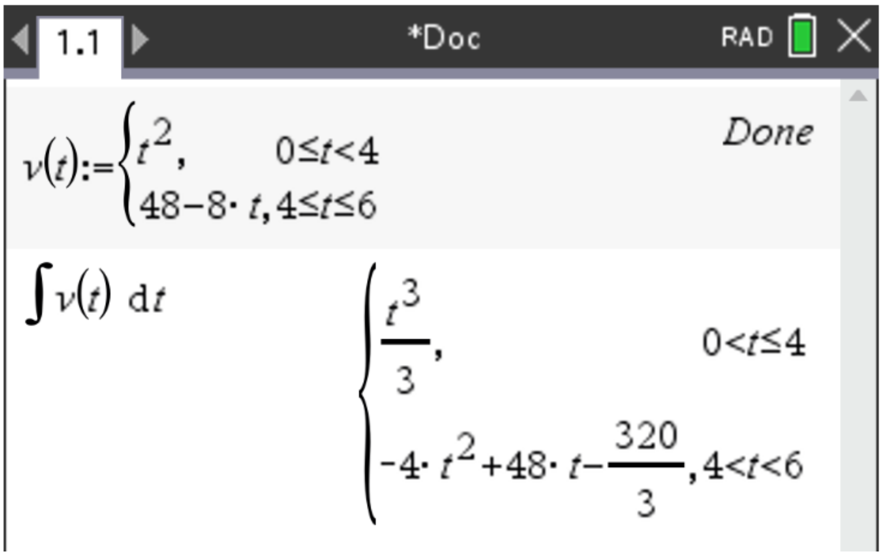

Figure 1: *TI-Nspire CAS* handheld computing an indefinite integral of a piecewise continuous function. Many students don't understand where does the number $-\frac{320}{3}$ come from.

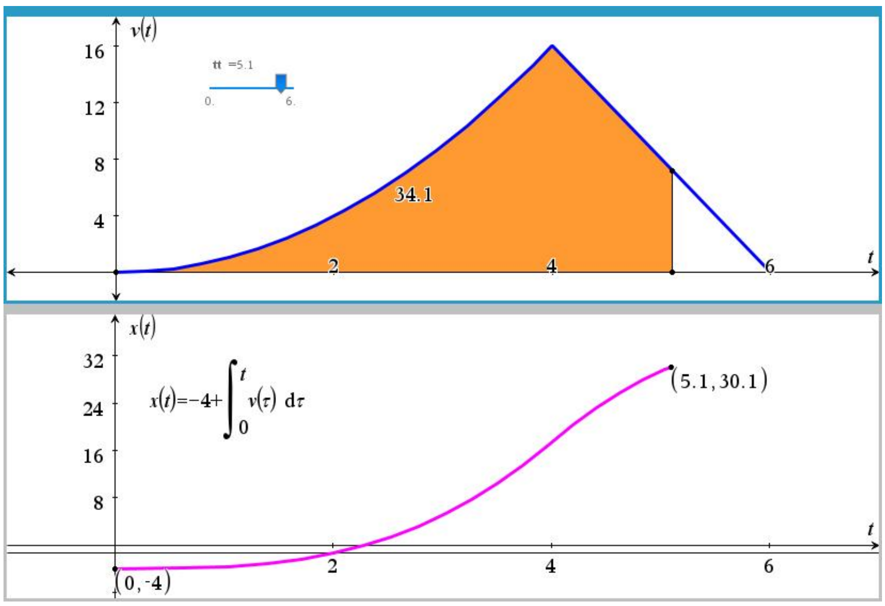

Figure 2: Animation of the FTC by *TI-Nspire CAS*. The function $v(t)$ is the one defined in Figure 1. The y-coordinate of the position $x(t)$ is 4 units less than the *area under the velocity curve* because the initial position is -4.

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