Applications of Computer Algebra (ACA) 2026 Conference.

Ovals and pedal curves: space trajectory models and genealogy of plane curves

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From news and classical curves to new curves

The general media frequently report on the launches of satellites and space probes, often accompanying these reports with graphics illustrating various orbital trajectories. These "general" representations serve as a rich foundation for sophisticated mathematical activities with a clear interdisciplinary STEAM approach. By networking between the graphical features of Dynamic Geometry Systems (DGS) and the symbolic strength of Computer Algebra Systems (CAS), we create an environment for exploration, conjecture validation, and rigorous modeling [5].

Following Kepler, central force motion is modeled using ellipses [2]. In this framework, a space probe's trajectory is represented by arcs of ellipses; changes in orbital geometry, specifically eccentricity, are induced by the activation of engines. Figure 1 illustrates this with the trajectory of SpaceIL Beresheet lunar probe.

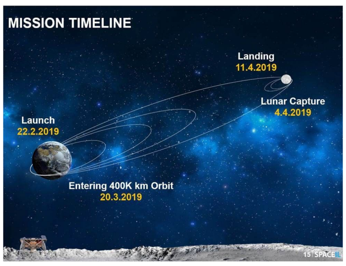

Figure 1: Beresheet trajectory to the Moon (NASA-ISA)

Historically, before the widespread adoption of Kepler's model, Giovanni Domenico Cassini proposed an alternative: the *Cassini ovals* (or spiric curves). Their bifocal definition is analogous to that of the ellipse:

- **Ellipse:** The locus of points M such that $MF_1 + MF_2 = r$.
- **Cassini Oval:** The locus of points M such that $MF_1 \cdot MF_2 = r$.

Recall that ellipses are reknown for their optical properties, and Cassini ovals appear as bisoptic curves of ellipses [7]. While an ellipse is a quadratic curve, a Cassini oval is determined by a quartic polynomial. It is an oval for sufficiently large r; otherwise, the spiric curve appears as two disjoint closed components that remain algebraically irreducible, the limiting case being a Bernoulli lemniscate. This requests advanced CAS features for algebraic verification; see [7, 8].

Let be given two points F_1 and F_2 in the plane and a positive real number r.

(i) Ellipse: The geometric locus of points M in the plane such that $MF_1 + MF_2 = r$ is an ellipse.

(ii) Cassini oval (spiric curve):The geometric locus of points M in the plane such that $MF_1 \cdot MF_2 = r$ is a Cassini oval.

Actually, it is possible to construct a Cassini oval from an ellipse using a transformation called inversion. Figure 2 enables a comparison of the shapes. Note the different values of r for the two curves in the figure.

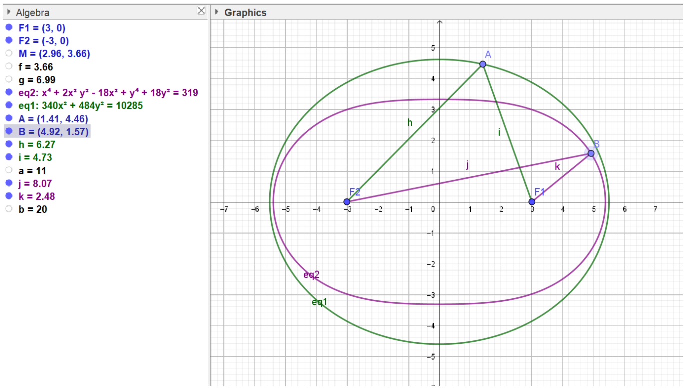

Figure 2: Visual comparison of an ellipse and a Cassini oval

Historically, another closed curve—the *Hippopede*—was conjectured as a model for planetary or satellite motion. This curve can be generated as the **pedal curve** of an ellipse with respect to its center [4].

Definition: Given a plane curve $\mathcal{C}$ and a point D (the pole), the pedal $\mathcal{P}$ of $\mathcal{C}$ with respect to D is the locus of the feet of the perpendiculars from D to the tangents of $\mathcal{C}$.

The pedal of an ellipse with respect to its center is known as a *Booth curve* or the *Hippopede of Proclus*. Distinguishing these curves from ellipses often requires high-precision visualization or symbolic analysis. Figure 3 illustrates this. On the left is displayed the construction of the pedal curve of the same ellipse appearing in Figure 2 with respect to the origin[1]. To distinguish between them, a strong zooming is needed. On the right, another construction is shown to make the difference clearer.

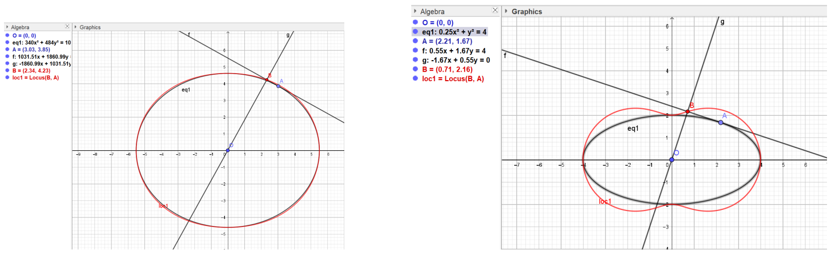

Figure 3: The Booth curve as pedal of an ellipse w.r.t. its center

The usage of software:

The modeling of these curves utilizes *GeoGebra-Discovery* [2]. While the numerical `Locus(B,A)` command provides immediate graphical feedback, the symbolic `LocusEquation(B,A)` command is essential for deriving the underlying polynomial equations. However, when symbolic commands encounter unsupported construction steps, we employ a hybrid algebraic approach. The way we can use to derive a polynomial equation for the pedal is as follows:

1. Derive a parametric (rational or trigonometric) representation of the base curve.

[1] With other poles, see [4].

[2] `https://github.com/kovzol/geogebra-discovery`

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2. Formulate the equations for the tangent and the corresponding perpendicular through the pole.

3. Use *Maple* and the `PolynomialIdeals` package to eliminate parameters via `EliminationIdeal`.

These methods have been applied in [4, 9]. If the initial parametric presentation of the ellipse was rational, what is obtained here is a rational parametrization. If the original was trigonometric, the new one is also trigonometric, but can be transformed into a rational parametrization (see [9]). Now a polynomial equation can be obtained as follows (we use Maple and its *PolynomialIdeals* package):

(i) Transform the rational parametrization into polynomials $P_1, P_2 \in \mathbb{R}[x, y, t]$.

(ii) These polynomials generate an ideal $I :=< P_1, P_2 >$.

(iii) Eliminate the parameter, using **EliminationIdeal**(I,x,y).

Another method consists in applying Maple's **implicitize** command, but this requests to guess the degree of the curve. Other methods exist also, e.g., using resultants (see [3]).

3D Extensions and Space Observation:

These 2D curves are often sections of 3D surfaces: ellipses originate from cones, while spiric curves originate from tori [7]. The *Hippopede of Eudoxus* serves as a spatial counterpart. Intriguingly, the projection of a geostationary satellite's orbit onto a plane is a hippopede[3]. Furthermore, these curves bear a topological resemblance to the *analemma*—the figure-eight path of the Sun observed from a fixed point over a year—though they remain mathematically distinct (compare the hippopede's equations with those given by [11]. A Hippopede of Eudoxus can be defined as follows:

Definition: Let be given a sphere $\mathcal{S}$. A hippopede is the geometric locus of a point on a great circle on $\mathcal{S}$ forming an angle α with the equatorial plane and turning at constant speed around the axis of the poles of the sphere, while the point itself travels on the great circle at the same speed in the opposite direction. The radius of the cylinder in which the hippopede is inscribed is then equal to $R\ \sin^2(\alpha/2)$. See [10, 2]. This result and a parametric presentation in 3D can be derived using "reverse engineering", as has been done in [7] for spiric curves. Figure 4 shows 3 snapshots of an animation obtained with Maple (the animation will be offered in the talk). Note that a hippopede is sometimes called also a spherical lemniscate. Because of its

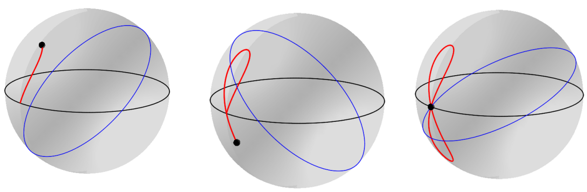

Figure 4: Creation of a hippopede of Eudoxus

shape and topology, it also "looks like" an analemma, which is the curve obtained when shooting a photo of the Sun from a fixed location on the Earth, every day at the same mean solar time, along one year. But they are different. Figure 5(a) shows the analemma in Hong Kong at 7:30pm and Figure 5(b) shows an analemma in Greenwich, both taken from [11].

[3]https://mathcurve.com/courbes3d.gb/hippopede/hippopede.shtml

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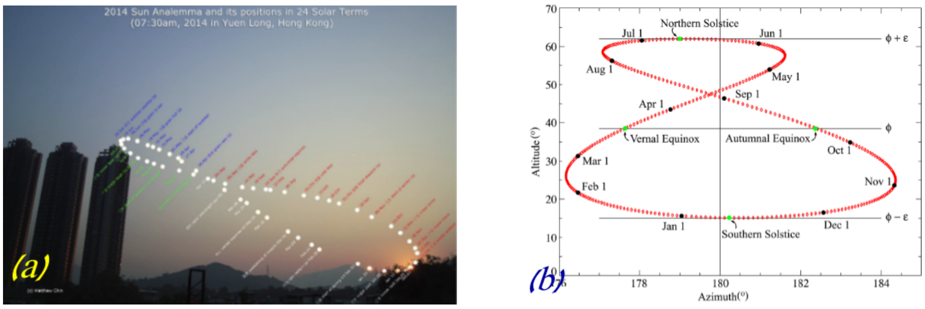

Figure 5: Analemma

Future Directions: The Genealogy of Curves

From ellipses, Cassini ovals are built by inversion and hippopedes of Proclus as pedals. Exploration of genealogy of curves is an illuminating endeavour. The study of the singularities of pedal curves remains a fertile ground for research [1]. By iteratively constructing "pedals of pedals," we generate sequences of curves with strictly increasing degrees. Exploring this "genealogy" through automated exploration of cubics, sextics, and octics provides a roadmap for future work in computational algebraic geometry; see [4, 8]. All this can be proposed to students at various levels, depending on their background in geometry and algebra.

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