Intersections of a Torus with Cubic Surfaces via Planar Section Reduction

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This paper studies the intersection between a torus of revolution and two related families of cubic surfaces: the structured class defined by a product of three affine linear factors perturbed by a constant, and the full cubic family written with twenty independent coefficients. The analysis is organised by horizontal sections $z = k$, which reduce the spatial problem to a family of planar problems between a cubic curve and the two circular components of the torus section. The investigation has a twofold purpose: first, to organise the mathematical analysis of torus–cubic intersections through section reduction; secondly, to extend the analysis from the factorised cubic family to the general cubic case and to document the results through paired symbolic and dynamic representations (Botana & Valcarce, 2002; Hohenwarter & Fuchs, 2005).

The torus and the section principle. Consider the torus $\mathcal{T}: (x^2 + y^2 + z^2 + R_m^2 - r_m^2)^2 - 4R_m^2(x^2 + y^2) = 0$, where $R_m > r_m > 0$. The rotational symmetry about the z-axis suggests the radial variable $Q = x^2 + y^2$. For a fixed section value $z = k$, the torus equation becomes quadratic in Q; its discriminant is $16R_m^2(r_m^2 - k^2)$. When $|k| < r_m$, the section decomposes exactly into two concentric circles with radii $\rho_\pm(k) = R_m \pm \sqrt{r_m^2 - k^2}$. When $|k| = r_m$, the two circles coalesce into $x^2 + y^2 = R_m^2$. This decomposition is exact, depends only on the quartic symmetry of the torus, and is independent of the cubic family chosen subsequently. This invariance is what renders the extension from a structured to a general cubic analytically manageable.

Two cubic families and the degree bound. The structured family $\mathcal{C}_\text{s}$ is defined by $L_1L_2L_3 + F_1 = 0$, where each $L_i(x, y, z) = A_ix + B_iy + C_iz + D_i$ is an affine linear form and F_1 is a perturbation constant. This class retains a geometric relation to three planes: when $F_1 = 0$ the cubic degenerates into their union; when $F_1 \neq 0$ factorisation vanishes from the zero set but the directional behaviour of the leading term persists. The general family $\mathcal{C}_\text{g}$ is defined by $P_3(x, y, z) = 0$, where P_3 is written in the standard monomial basis of degree at most three with twenty independent coefficients $A_1, \ldots, A_{20}$. This family is no longer constrained by factorisation into linear forms.

For both families, substituting $z = k$ yields a planar cubic. Since each torus-section component is a conic of degree two, Bézout's theorem gives an upper bound of $2 \times 3 = 6$ intersection points (counted with multiplicity) on each circle, provided the cubic and the circle share no component. When both torus circles are present and all intersections are real and transverse, a single horizontal section may contain twelve real intersection points. The passage from $\mathcal{C}_\text{s}$ to $\mathcal{C}_\text{g}$ changes the algebraic source of the cubic behaviour but preserves both the section framework and this degree-theoretic bound.

Exploration procedure and worked examples. The analytical workflow proceeds through a pseudo-algorithm valid for both families: fix the torus parameters and the cubic coefficients; choose k with $|k| \leq r_m$; substitute $z = k$ in the cubic equation; compute the torus radii $\rho_\pm(k)$; solve the two planar systems (cubic against each circle); classify solutions; lift each real planar solution to the spatial point (x, y, k); and repeat for a sequence of section values to detect persistence, tangency, or disappearance of real solutions (Recio et al., 2022).

A structured reference sample with $R_m = 4$, $r_m = 1$, and three specified linear factors was examined at $k = 0.529$, where both torus circles are met by the planar cubic in six real points, attaining the maximal count. A general example was then constructed with twenty distinct integer coefficients, $(A_1, \ldots, A_{20}) = (10, 1, 5, -4, 12, -5, -6, 2, 9, 7, 3, -2, 11, -7, -11, -1, -3, -9, -8, -10)$, and the same torus parameters. Three representative section levels ($k = 0$, $k = 0.5$, $k = 0.9$) were examined. In each case, the planar cubic meets both torus circles in six real points, confirming that the maximal intersection count does not depend on factorisation into three planes but persists in a fully general coefficient regime.

Symbolic and dynamic complementarity. The complementary use of symbolic computation and dynamic geometry is central to the methodology. Symbolic environments support substitution, radial rewriting, identification of admissible section levels, and exact or numerical computation of intersection points (Botana & Valcarce, 2002). Dynamic environments reorganise the same structure as a family of related representations: the three-dimensional torus, the moving plane $z = k$, the two torus circles, the planar cubic section, and the lifted intersection points (Hohenwarter & Fuchs, 2005; Recio et al., 2022). This complementarity is consistent with research relating dynamic geometry environments to conjecturing, proving, and movement across representations (Komatsu & Jones, 2020). The symbolic environment ensures control of algebraic form, while the dynamic environment renders regime changes visible. Their joint use distributes mathematical labour rather than duplicating it.

Conclusions. The horizontal-section method provides an effective framework for studying torus–cubic intersections. For the torus, it yields an exact decomposition into two concentric circles. For the cubic, the same principle produces a planar cubic regardless of whether the spatial equation is factorised or fully general. The extension from the structured to the general family preserves the section method, the Bézout bound, and the symbolic–dynamic complementarity, whilst changing the geometric interpretation. The worked example with twenty distinct integer coefficients demonstrates that maximal real intersection behaviour persists in the general regime, confirming that the phenomena observed in the planar reduction belong to the interaction between cubic section families and toroidal section structure rather than to factorisation alone. The problem supports several forms of mathematical activity—symbolic derivation, comparison of structured and general families, numerical localisation, and interpretation of invariant features—offering a coherent environment for advanced undergraduate or postgraduate study (Dolgachev, 2012).

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