

Solving Wasan Problems on Malfatti Circles via the *mnr* Method: Human-AI Collaborative Automated Derivation using KeTCindy

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1 Introduction

During the Edo period (from the 17th to the mid-19th century) in Japan, interactions with Europe were strictly limited due to the national isolation policy (*Sakoku*). As a consequence, a unique, indigenous mathematical tradition known as *Wasan* flourished independently. *Wasan* mathematicians established advanced methods for solving complex algebraic equations, which enabled them to solve a wide variety of intricate geometric problems. One notable example is the problem of Malfatti circles. For instance, it is depicted as the second figure from the left on a *Sangaku* [2] dedicated at Myojo-rinji Temple in Gifu Prefecture in 1865.

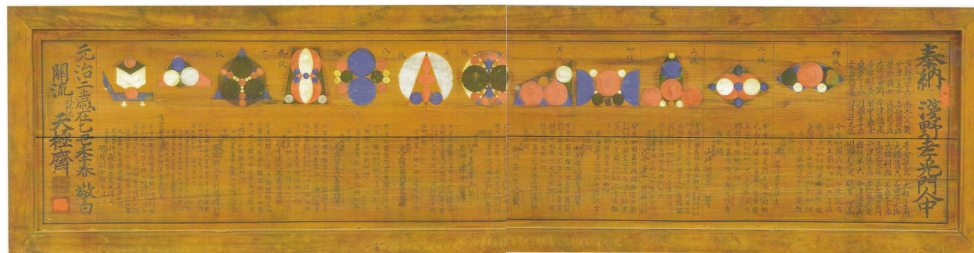


Fig.1 Malfatti circles depicted on a *Sangaku*

Malfatti circles were studied by the prominent *Wasan* mathematician Naonobu Ajima earlier than Gian Francesco Malfatti, and this study was featured in a book published posthumously (*Fukyu Sanpo*, 1799). However, Ajima’s problem asked to determine the radius¹ of each circle given the three side lengths of a triangle. In contrast, the Myojo-rinji Temple problem requires finding the radius r of the triangle’s incircle given the radii r_1, r_2, r_3 of the three interior circles. Kazuhide Omura provided the following solution to this problem in his 1841 publication, *Sanpo Tenzan Tebikigusa* [4]:

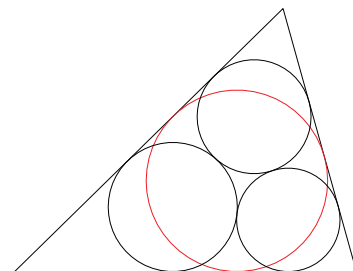


Fig.2 Malfatti Circles

$$r = \frac{\sqrt{r_1 r_2 r_3} (\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3})}{\sqrt{r_1 r_2} + \sqrt{r_1 r_3} + \sqrt{r_2 r_3}}$$

While *Wasan* mathematicians solved these problems relying on their extraordinary intuition and computational skills, this paper aims to solve the Malfatti circles problem from the Myojo-rinji Temple *Sangaku* using modern computer algebra systems (CAS). Specifically, formulating a system of equations based directly on the tangency conditions of the circles typically introduces irrational expressions, quickly leading to computational failure. To overcome this, we employ a method that expresses various geometric quantities as rational expressions in terms of the tangents of the two half base angles, $m = \tan(B/2)$ and $n = \tan(C/2)$, along with the inradius r . This approach is referred to as the *mnr* method.

¹In *Wasan*, the dimensions of circles were typically given as diameters, but we use radii in this paper.

2 Formulation by *mnr* Method

KeTCindy[1] is a script library for the dynamic geometry software Cinderella, originally developed to generate \TeX drawing codes. Subsequently, features were added to internally invoke Maxima, R, and GCC, enabling users to utilize their computational results. `mnr.max` is a Maxima library that can be loaded via command `batch`. In KeTCindy, the following commands are formulated and executed. **Fig.3** displays the Cinderella interface after executing `cmd1`.

```
"cmdL1=concat(Mxbatch("mnr"),[,
"putT(m,n,r); A:vtxT; B:vtxL; C:vtxR; I:inC",
"putT(m,n,s1*r); slideT(vtxL,B); D:vtxT; Eb:vtxR; I1:inC",
"putT(m,n,s2*r); slideT(vtxR,C); G:vtxT; Ec:vtxL; I2:inC",
"putT(m,n,s3*r); slideT(vtxT,A); H:vtxL; J:vtxR; I3:inC",
"eq1:numer(lenSeg2(I1,I2)-(s1*r+s2*r)\hats 2)",
"eq2:numer(lenSeg2(I2,I3)-(s2*r+s3*r)\hats 2)",
"eq3:numer(lenSeg2(I3,I1)-(s3*r+s1*r)\hats 2)",
"eq1:eq1/r^2; eq2:eq2/r^2; eq3:eq3^2"
]);
var1="A::B::C::I::D::Eb::G::Ec::H::J::I1::I2::I3::eq1::eq2::eq3";
```

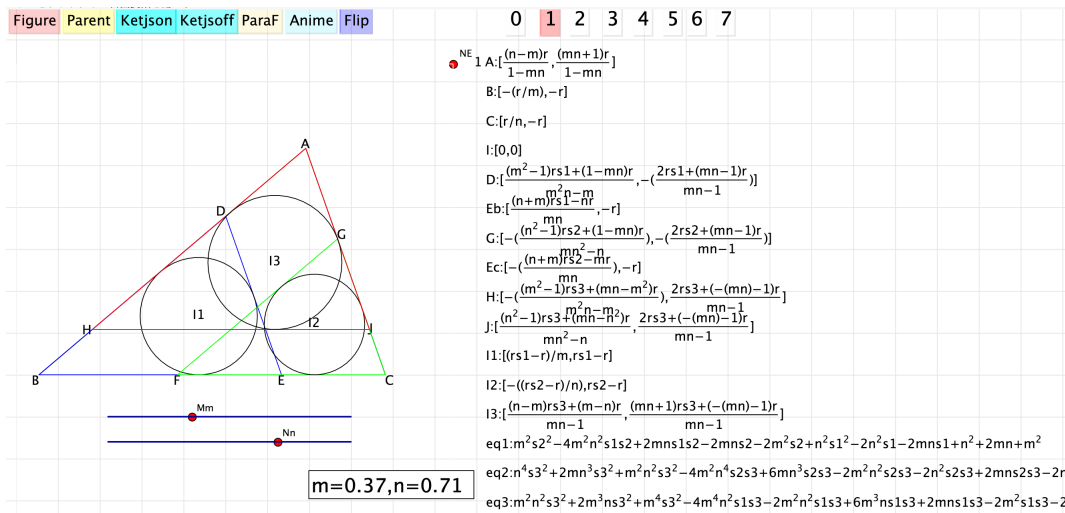


Fig.3 Screenshot of the Cinderella interface after executing the commands

3 Human-AI Collaborative Scripting

The next step is to solve the system of equations `eq1`, `eq2`, and `eq3` obtained in the previous section for $s_j = r_j/r$ ($j = 1, 2, 3$). However, when writing scripts for KeTCindy and Maxima, several practical issues arise:

- The scripts themselves become lengthy and difficult to read.
- The assignment operators differ between the two systems (namely, `=` in Cinderella and `:` in Maxima), leading to frequent typographical errors.
- It is difficult to isolate and identify runtime errors generated by Maxima.
- In the string-based command list `cmdL`, it is easy to mistype line endings (e.g., confusing commas, semicolons, or double quotes) and forget to remove the trailing comma on the last element of `cmdL`.

These factors make debugging and identifying errors extremely difficult. To overcome these scripting challenges, we decided to proceed by interactively collaborating with the generative AI, Gemini.

4 Overcoming Computational Explosion with AI

Even after rationalizing the equations using the *mnr* method, attempting to solve them directly with standard Gröbner basis computations in Maxima results in a severe computational explosion. Through interactive consultations with Gemini, we shifted our strategy from relying

solely on Gröbner bases to utilizing elimination methods (namely, the `eliminate` command or `reduceD`).

Furthermore, we incorporated human geometric intuition into the automated process. By focusing on the inherent symmetry of the Malfatti circles, we refined the variable substitutions, introducing quarter-angle tangents $M = \tan(B/4)$ and $N = \tan(C/4)$ alongside symmetric polynomials. This “human-in-the-loop” synergy successfully guided the CAS to factorize the complex expressions step-by-step, effectively bypassing the computational bottleneck.

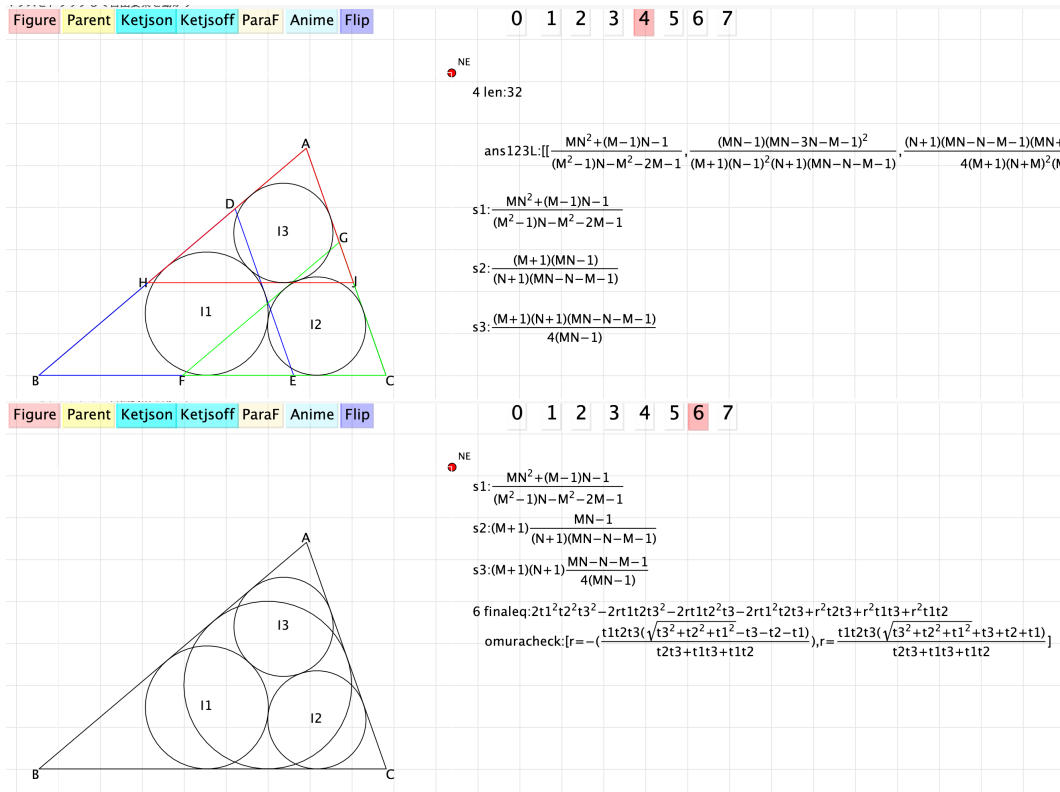


Fig.4 Screenshot of step4 and step6

5 Conclusion

By integrating the historical wisdom of *Wasan*, the geometric algebraic formulation of the *mnr* method, and the computational power of KeTCindy and Maxima, we successfully re-derived Kazuhide Omura’s beautiful 1841 formula.

A crucial factor in this success was the collaborative problem-solving with generative AI. Gemini acted not merely as a code debugger, but as a strategic partner in navigating algorithmic limitations. This study highlights that human-AI collaboration holds immense potential for the future of computer algebra research and mathematics education, enabling us to unravel complex problems that were previously computationally intractable.

References

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