Weak Turbulent Kolmogorov Spectrum for Surface Gravity Waves

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We study the long-time evolution of surface gravity waves on deep water excited by a stochastic external force concentrated in moderately small wave numbers. We numerically implemented the primitive Euler equations for the potential flow of an ideal fluid with free surface written in Hamiltonian canonical variables, using the expansion of the Hamiltonian in powers of nonlinearity of terms up to fourth order. We show that because of nonlinear interaction processes a stationary Fourier spectrum of a surface elevation close to $\langle|\eta_k|^2\rangle \sim k^{-7/2}$ is formed. The observed spectrum can be interpreted as a weak-turbulent Kolmogorov spectrum for a direct cascade of energy.

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Kolmogorov was born in 1903. Now, in the year of his centenary, his greatness is obvious not only for pure and applied mathematicians but also all physicists appreciate his pioneering works on powerlike cascade spectra in turbulence of the incompressible fluid [1]. It is obvious now that cascade processes, similar to the Kolmogorov cascade of energy, play a very important role in many different fields of physics, such as nonlinear optics [2], plasma physics [3], hydrodynamics of superfluid He^4, and so forth.

In all these cases the physical situations are similar. There is an ensemble of slowly decaying, weakly nonlinear waves in a medium with dispersion. Such systems have to be described statistically. However, this is not traditional statistical mechanics, because the ensembles are very far from thermodynamic equilibrium. Nevertheless, one can develop a systematic approach for the statistical study of weakly nonlinear waves. This is the theory of weak (or wave) turbulence [4]. The main tools here are the kinetic equations for squared wave amplitudes. These equations describe the nonlinear resonant interaction processes taking place in the wave systems. As in the turbulence in incompressible fluid, these processes lead to the cascades of some constants of motion (energy, wave action, momentum, etc.) along the k space. In isotropic systems it might be either a direct cascade of energy from small to large wave numbers or an inverse cascade of wave action to small wave numbers [5]. In an anisotropic system the situation could be much more complicated [6].

The brilliant conjecture of Kolmogorov still is a hypothesis, supported by ample experimental evidence. On the contrary, the existence of powerlike Kolmogorov spectra, describing cascades in weak turbulence, is a rigorous mathematical fact. These spectra are the exact solutions of the stationary homogeneous kinetic equation, completely different from the thermodynamic Rayleigh-Jeans solutions.

Nevertheless, the case is not closed. The weak-turbulent theory itself is based on some assumptions, like phase stochasticity and the absence of coherent structures. This is the reason why justification of weak-turbulent theory is an urgent and important problem.

This justification can be done by a direct numerical solution of the primitive dynamic equation describing the wave ensemble. In pioneering works by Majda, McLaughlin, and Tabak [7] it was done for the 1D wave system. The results obtained by these authors are not easily interpreted. In some cases they demonstrate Kolmogorov-type spectra, and in other cases power spectra with essentially different exponents.

In Ref. [8] deviation from weak-turbulent theory was explained by the role of coherent structures (solitons, quasisolitons, and collapses). If a 1D system is free from coherent structures, weak-turbulent spectra are observed with a good deal of evidence [9–11].

In spite of their heuristic value, the 1D models so far developed have no direct physical application. Real physical systems, where wave turbulence is realized, are at least two dimensional. The most natural and important examples are capillary and gravity waves on deep water. A weak-turbulent theory of capillary waves was developed by Zakharov and Filonenko in 1967 [12], who found that the correlation function of elevation $\eta(\vec{r}, t)$ has to be $\langle|\eta_{\vec{k}}|^2\rangle \sim k^{-19/4}$. This result was supported by laboratory experiments, performed independently by three groups (in UCLA [13], Niels Bohr Institute [14], and the Solid State Physics Institute in Chernogolovka, Russia [15,16]). The spectrum $k^{-19/4}$ was obtained by a direct numerical simulation of Euler equation for incompressible fluid with free surface by Pushkarev and Zakharov [17–19].

The most interesting example of 2D wave ensembles demonstrating weak-turbulent cascades is a system of gravity waves on the surface of deep water. We are sure that a weak-turbulent theory of these waves is key to understanding the processes in a wind-driven sea. However, we do not concentrate on this point in this Letter.

Our initial goal was to reproduce (and emulate), for gravity waves, the work that was done by Pushkarev and Zakharov [17] for capillary waves. One has to expect that this is a more difficult problem, because the leading process in capillary waves is a three-wave interaction (dispersion of waves is of decay type), while for gravity waves the lowest order process is four-wave interaction (dispersion of waves is of nondecay type).

Attempts to perform direct numerical simulations of potential flow in an ideal fluid with a free surface were made by several authors [20]. In only one paper authors did pay attention to Kolmogorov-type weak-turbulent spectra [21]. Authors of this paper observed the formation of Kolmogorov tails during the time evolution of an artificially cut-off JONSWAP (Joint North Sea Wave Project, described in [22]) energy spectrum. The results of this Letter agree with the results of [21] completely. However, we stress a difference.

In our work, in contrast to [21], we study a forced turbulence, excited by external sources, posed in moderately low wave numbers. We show that the growth of wave energy due to this forcing is arrested by nonlinear resonant four-wave processes, which leads to the formation of a powerlike Kolmogorov spectrum in the inertial interval. In this sense our Letter is a direct numerical confirmation of the weak-turbulent theory for surface gravity waves.

Theoretical background.—We study the potential flow of an ideal inviscid incompressible fluid with the velocity potential $\phi = \phi(x, y, z; t)$

$$\Delta\phi = 0,$$

in the infinitely deep domain occupied by the fluid. Equations for the boundary conditions at the surface are the following:

$$\begin{aligned} (\dot{\eta} + \phi'_x \eta'_x + \phi'_y \eta'_y)|_{z=\eta} &= \phi'_z|_{z=\eta}, \\ (\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2)|_{z=\eta} + g\eta &= 0. \end{aligned} \tag{1}$$

Here $\eta(x, y; t)$ is the surface elevation with respect to still water, and g is the gravity acceleration. Equations (1) are Hamiltonian [23] with the canonical variables $\eta(x, y; t)$ and $\psi(x, y; t) = \phi(x, y, \eta(x, y; t); t)$

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \tag{2}$$

where H is the Hamiltonian of the system

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} dxdy\left(g\eta^2 + \int_{-\infty}^{\eta} |\nabla\phi|^2 dz\right).$$

Unfortunately H cannot be written in the close form as a functional of η and ψ. However, one can limit Hamiltonian by the first three terms of powers of η and ψ

$$\begin{aligned} H &= H_0 + H_1 + H_2 + \dots, \\ H_0 &= \frac{1}{2}\int (g\eta^2 + \psi\hat{k}\psi)dxdy, \\ H_1 &= \frac{1}{2}\int \eta[|\nabla\psi|^2 - (\hat{k}\psi)^2]dxdy, \\ H_2 &= \frac{1}{2}\int \eta(\hat{k}\psi)[\hat{k}(\eta(\hat{k}\psi)) + \eta\Delta\psi]dxdy. \end{aligned} \tag{3}$$

Here $\hat{k}$ is a linear integral operator ($\hat{k} = \sqrt{-\Delta}$), such that in k space it corresponds to multiplication of Fourier harmonics ($\psi_{\vec{k}} = \frac{1}{2\pi}\int \psi_{\vec{r}} e^{i\vec{k}\vec{r}} dxdy$) by $\sqrt{k_x^2 + k_y^2}$. For gravity waves this reduced Hamiltonian describes four-wave interaction. Then dynamical Eqs. (2) acquire the form

$$\begin{aligned} \dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) \\ &\quad + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi], \\ \dot{\psi} &= -g\eta - \tfrac{1}{2}[(\nabla\psi)^2 - (\hat{k}\psi)^2] - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] \\ &\quad - [\eta\hat{k}\psi]\Delta\psi + D_{\vec{r}} + F_{\vec{r}}. \end{aligned} \tag{4}$$

Here $D_{\vec{r}}$ is a dissipation term, which consists of hyperviscosity on small scales and "artificial" damping on large scales. $F_{\vec{r}}$ is the driving term that simulates pumping on large scales (for example, due to wind). In the k space supports of $D_{\vec{k}}$ and $F_{\vec{k}}$ are separated by the inertial interval, where the Kolmogorov-type solution can be recognized.

Another approach based on boundary integral approximation has been studied by Clamond and Grue in [24].

We study numerically the quasistationary solution of Eqs. (4). According to the theory of weak turbulence, the Fourier spectrum of the surface elevation averaged by ensemble (corresponding to the flux of energy from large scales to small scales) is

$$\langle|\eta_k|^2\rangle = \frac{Cg^{1/2}P^{1/3}}{k^{7/2}}. \tag{5}$$

Here P is the energy flux, and C is a dimensionless Kolmogorov constant. This spectrum is a stationary solution of the wave kinetic equation [12].

It is worth noting that an alternative spectrum was proposed earlier by Phillips [25]. In contrast to weak-turbulent theory, the powerlike spectrum of Phillips can be formed due to a wave breaking mechanism and leads to a surface elevation spectrum $\langle|\eta_k|^2\rangle \sim k^{-4}$. It can be realized for a higher level of turbulence, which is far beyond the weak-turbulent theory.

Numerical simulation.—For numerical integration of (4) we used the following pumping and dissipation terms, which are defined in Fourier space as

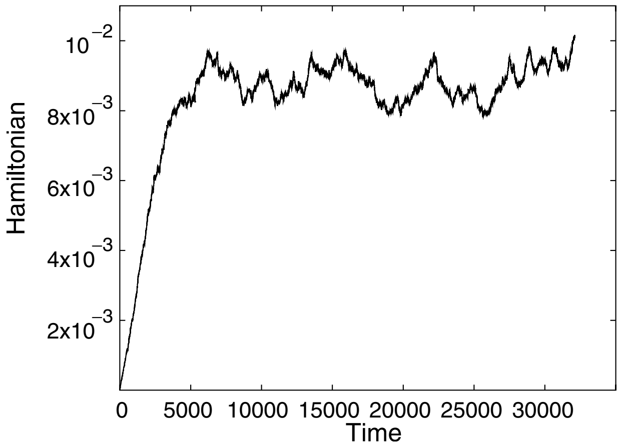

FIG. 1. Hamiltonian as a function of time. Quasistationary state is formed at $t \simeq 5000$.

$$F_k = f_k e^{iR_{\vec{k}}(t)},$$
$$f_k = \begin{cases} 4F_0 \frac{(k-k_{p1})(k_{p2}-k)}{(k_{p2}-k_{p1})^2}, & \text{if } k_{p1} < k < k_{p2}, \\ f_k = 0, & \text{otherwise,} \end{cases}$$
$$D_{\vec{k}} = \gamma_k \psi_{\vec{k}},$$
$$\gamma_k = \begin{cases} -\gamma_1, & k \le k_{p1}, \\ 0, & k_{p1} < k < k_{p2}, \\ -\gamma_2 (k - k_d)^2, & k > k_d. \end{cases} \tag{6}$$

Here $R_{\vec{k}}(t)$ is a uniformly distributed random number in the interval $(0, 2\pi)$ for each $\vec{k}$. We have applied an implicit difference scheme that keeps the main property of this system—conservation of the Hamiltonian in the absence of pumping and damping.

Equations (4) were numerically simulated in the periodic domain $2\pi \times 2\pi$. The size of the grid was 512×512 points. Gravity acceleration g was equal to one. Parameters of the damping and pumping in (6) were the following: $k_{p1} = 5$, $k_{p2} = 10$, and $k_d = 100$. Thus, the inertial interval is equal to a decade.

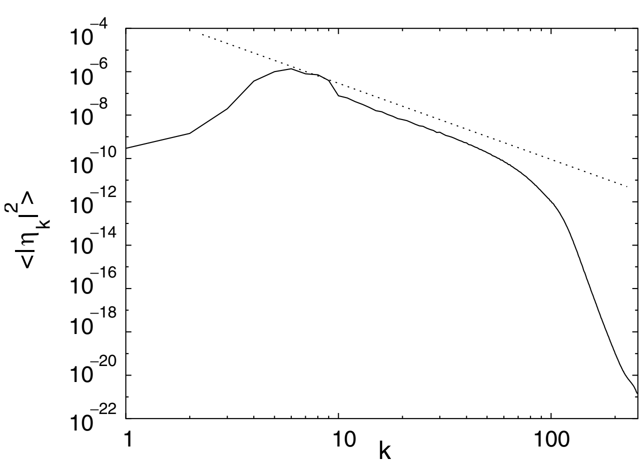

FIG. 2. Averaged spectrum of surface elevation $\langle |\eta_k|^2 \rangle$. Line $\sim k^{7/2}$ is also shown.

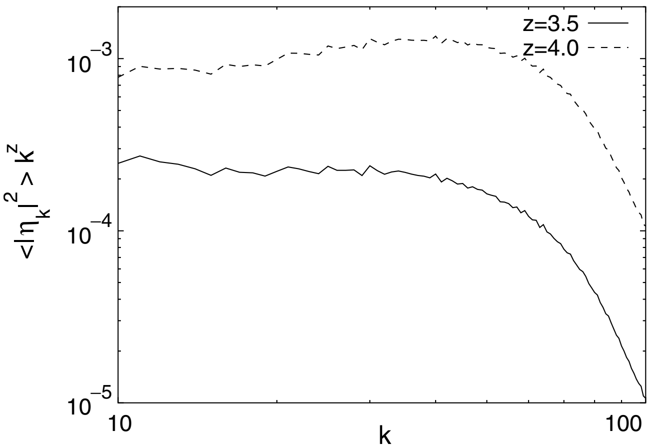

FIG. 3. Compensated waves spectra. Solid line, weak turbulent theory; dashed line, Phillips theory. Only one decade in k is shown.

In the simulations we paid special attention to the problems that could "damage" the calculations. First of all, it is the "bottleneck" phenomenon that was studied in Ref. [26]. The effect consists in accumulation of energy in k space ahead of the dissipation region, if the dissipation is too large. This effect is very fast, but can be effectively suppressed by a proper choice of damping value γ_2 in dissipation (6) in the case of moderate pumping values F_0. On the other hand, the F_0 value should not be too small to secure a wave cascade on the discrete grid. For the case of capillary waves it was examined in detail in [27]. The second problem is the accumulation of "condensate" in low wave numbers due to inverse cascade. Buildup of condensate can be overcome by simple adaptive damping in the small wave numbers.

After some time the system reaches the stationary state, where the balance between pumping and damping takes place. In this state an important parameter is the ratio of nonlinear energy to the linear one $(H_1 + H_2)/H_0$.

For example, for the external force $F_0 = 2 \times 10^{-4}$, $\gamma_1 = 1 \times 10^{-3}$, $\gamma_2 = 665$ the level of nonlinearity was

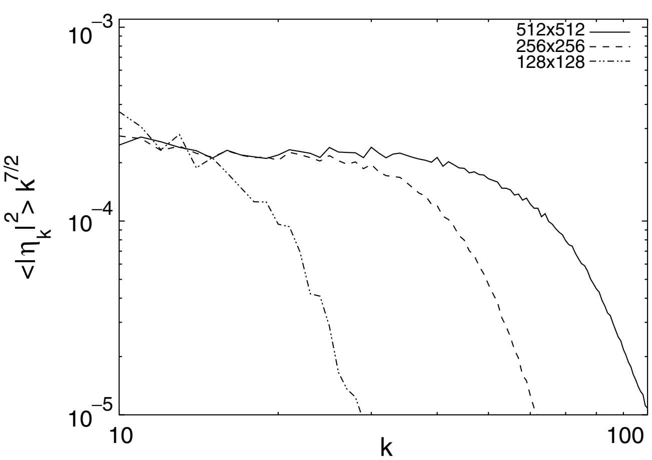

FIG. 4. Compensated spectra for the different grids.

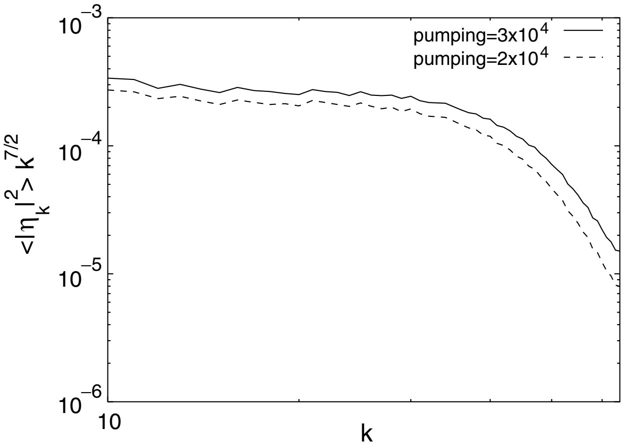

FIG. 5. Compensated spectra for the different pumping levels. Solid line, $F_0 = 3 \times 10^{-4}$; dashed line, $F_0 = 2 \times 10^{-4}$.

equal to $(H_1 + H_2)/H_0 \simeq 3 \times 10^{-3}$. The Hamiltonian as a function of time is shown in Fig. 1.

The averaged spectrum of surface elevation $\langle|\eta_k|^2\rangle$ appears to be powerlike in the essential part of the inertial interval, where the influence of pumping and damping was small. This spectrum is shown in Fig. 2.

One can estimate the exponent of the spectrum. Compensated spectra (e.g., multiplied by k^z to get horizontal line) are shown in Fig. 3. This figure seems to be the evidence that the Kolmogorov spectrum predicted by weak turbulence theory fits better the results of the numerical experiment.

The quality of the result (closeness to $\langle|\eta_k|^2\rangle \sim k^{-7/2}$) crucially depends on the width of the inertial interval. In our previous work [28] similar simulations were performed on the grid 256×256. Weak-turbulent spectrum is clearly seen on the grid 512×512, can be hinted at the grid 256×256, and is almost invisible on the grid 128×128. This difference is demonstrated in Fig. 4.

In conclusion, we note that for the different pumping level surface elevation spectra differ only due to the different energy flux value P in (5), as clearly seen in Fig. 5.

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