Simultaneous Numerical Simulation of Direct and Inverse Cascades in Wave Turbulence

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The results of the direct numerical simulation of isotropic turbulence of surface gravity waves in the framework of Hamiltonian equations are presented. For the first time, the simultaneous formation of both direct and inverse cascades has been observed in the framework of the primordial dynamical equations. At the same time, a strong long wave background has been developed. It has been shown that the Kolmogorov spectra obtained are very sensitive to the presence of this condensate. Such a situation has to be typical for experimental wave tanks, flumes, and small lakes.

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Introduction.—This year is the 50th anniversary of the famous work by Phillips [1] which was, probably, the first attempt to give an explanation for powerlike spectra of surface gravity waves observed in numerous experiments. In recent works [2,3], the physical explanation given by Phillips has been corrected. Within less than ten years of the publication of Phillips' paper, the statistical theory of surface waves was founded: Hasselmann derived the kinetic equation for waves [4], and Zakharov created the theory of wave (or weak) turbulence [5] which describes the solutions of this equation. Stationary Kolmogorov solutions of the kinetic equation corresponding to a flux of energy from large to small scales (direct cascade) and a flux of wave action (wave number) from small to large scales (inverse cascade) were found [5,6]. This opened a way for the creation of an effective tool for wave forecasting. The conjectured assumptions with which the theory of weak turbulence was derived include Gaussian statistics for the wave field and the prevalence of resonant interactions [5]. These assumptions are subject to confirmation.

Experiments in the open sea and on the Great Lakes gave temporal and space spectra consistent with the theory [7–12]. Most of these experiments were performed with wind pumping, which is broad in spectrum. Narrow in spectrum pumping can be realized in wave tanks or flumes. The results obtained on such state-of-the-art devices frequently contradict predictions of the theory of wave turbulence. For example, in the recent experiments of Refs. [13,14], the observed spectra changed slope with variation of steepness and pumping force.

Perhaps the most promising way to check the conjectures of the wave turbulence theory is via numerical experiment. In the case of direct numerical simulation, we have the highest possible control of the parameters of experiments and can access all information about the wave field. This wealth of data is available at the cost of enormous computational complexity. The rapid growth of computational power and development of efficient computational algorithms has allowed direct numerical simulation of the surface gravity waves, starting from the simulations of swell evolution [15–20] to isotropic turbulence simulation [21–24]. There is hope that this approach together with confirmation of the conjectures of weak turbulent theory will allow us to explain phenomena observed in experimental wave tanks.

The theory of the wave turbulence is still under development. To close the circle, the recent paper by Newell and Zakharov [3] gave a second life to the Phillips spectrum, although from a completely different point of view. The Phillips spectrum is considered to be a solution which gives a balance of the transfer of energy due to nonlinear wave interaction and transfer due to intermittent events such as wave breaking and whitecapping.

This Letter was inspired by several recent papers. In the first one [24], the numerical simulation of isotropic turbulence with observed formation of an inverse cascade was performed in the framework of the Zakharov equations [5]. A little bit later, a group of authors [25] during simulation of 2D hydrodynamics observed the formation of large scale structure due to Kraichnan's inverse cascade and explored its influence on the system. Influence of the condensate on turbulence in plasma was simulated recently in Ref. [26]. Approximately at the same time, a state-of-the-art surface wave experiment was performed [14]. The observed spectra differed from the theory of wave turbulence.

In this Letter, the results of a direct numerical simulation of isotropic turbulence of surface gravity waves in the framework of Hamiltonian equations are reported. For the first time, the formation of both direct and inverse cascades was observed in the framework of the primordial dynamical equations. At the same time, a strong long wave background was developed. This phenomenon of "condensation" [27] of waves (following the analogy with Bose-Einstein condensation in condensed matter physics) was predicted by the theory of weak turbulence. It was shown that the Kolmogorov spectra obtained are very sensitive to the presence of the condensate. Such a situation has to be typical for experimental wave tanks, flumes, and small lakes. These results can be considered as the first observation of generalized Phillips spectra introduced in Ref. [3] and explain some deviations from the wave turbulence theory in recent wave tank experiments.

Theoretical background.—We consider a potential flow of ideal incompressible fluid. The system is described in terms of weakly nonlinear equations [5,22] for surface elevation $\eta(\vec{r}, t)$ and velocity potential at the surface $\psi(\vec{r}, t)$ $[\vec{r} = \overrightarrow{(x, y)}]$

$$\begin{aligned} \dot{\eta} &= \hat{k}\psi - [\nabla(\eta\nabla\psi)] - \hat{k}[\eta\hat{k}\psi] + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) \\ &\quad + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi] + \hat{F}^{-1}[\gamma_k\eta_k], \\ \dot{\psi} &= -g\eta - \tfrac{1}{2}[(\nabla\psi)^2 - (\hat{k}\psi)^2] - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] \\ &\quad - [\eta\hat{k}\psi]\Delta\psi - \hat{F}^{-1}[\gamma_k\psi_k] + \hat{F}^{-1}[P_{\vec{k}}]. \end{aligned} \tag{1}$$

Here the dot means time derivative, Δ is the Laplace operator, $\hat{k}$ is a linear integral operator ($\hat{k} = \sqrt{-\Delta}$), $\hat{F}^{-1}$ is an inverse Fourier transform, and γ_k is a dissipation rate (according to recent work [28], it has to be included in both equations), which corresponds to viscosity on small scales and, if needed, "artificial" damping on large scales. $P_{\vec{k}}$ is the driving term which simulates pumping on large scales (for example, due to wind). In the k space, supports of γ_k and $P_{\vec{k}}$ are separated by the inertial interval, where the Kolmogorov-type solution can be recognized.

In the case of statistical description of the wave field, the Hasselmann kinetic equation [4] for the distribution of the wave action $n(k, t) = \langle |a_{\vec{k}}(t)|^2 \rangle$ is used. Here

$$a_{\vec{k}} = \sqrt{\omega_k/(2k)}\eta_{\vec{k}} + i\sqrt{k/(2\omega_k)}\psi_{\vec{k}} \tag{2}$$

are complex normal variables. For gravity waves, $\omega_k = \sqrt{gk}$. From the theory of weak turbulence [5,6], besides the equipartions (Rayleigh-Jeans) spectrum, we know two stationary solutions of the kinetic equation in the case of four-wave interaction:

$$\begin{aligned} n_k^{(1)} &= C_1 P^{1/3} k^{-(2\beta/3)-d}, \\ n_k^{(2)} &= C_2 Q^{1/3} k^{-[(2\beta-\alpha)/3]-d}. \end{aligned} \tag{3}$$

For surface gravity waves, the coefficient of homogeneity of the nonlinear interaction matrix element $\beta = 3$, the power of the dispersion law $\alpha = 1/2$, and the dimension of the surface $d = 2$. As a result, we get

$$n_k^{(1)} = C_1 P^{1/3} k^{-4}, \qquad n_k^{(2)} = C_2 Q^{1/3} k^{-23/6}. \tag{4}$$

The first solution $n_k^{(1)}$ describes the direct cascade of energy, and the second solution $n_k^{(2)}$ describes the inverse cascade of action.

Numerical simulation.—We simulated the primordial dynamical equations (1) in a periodic spatial domain $2\pi \times 2\pi$. The main part of the simulations was performed on a grid consisting of 1024×1024 nodes. Also, we performed a long time simulation on a 256×256 grid. The numerical code used was verified in Refs. [18–22,29]. Gravity acceleration was $g = 1$. The pseudoviscous damping coefficient had the following form:

$$\gamma_k = \begin{cases} \gamma_0(k - k_d)^2, \\ 0 & \text{if } k \le k_d, \end{cases} \tag{5}$$

where $k_d = 256$ and $\gamma_{0,1024} = 2.7 \times 10^4$ for the 1024×1024 grid and $k_d = 64$ and $\gamma_{0,256} = 2.4 \times 10^2$ for the smaller 256×256 grid. Pumping was an isotropic driving force narrow in the wave number space with a random phase:

$$\begin{aligned} P_{\vec{k}} &= f_k e^{iR_{\vec{k}}(t)}, \\ f_k &= \begin{cases} 4F_0 \frac{(k-k_{p1})(k_{p2}-k)}{(k_{p2}-k_{p1})^2}, \\ 0 & \text{if } k < k_{p1} \text{ or } k > k_{p2}; \end{cases} \end{aligned} \tag{6}$$

here $k_{p1} = 28$, $k_{p2} = 32$, and $F_0 = 1.5 \times 10^{-5}$; $R_{\vec{k}}(t)$ was a uniformly distributed random number in the interval $(0, 2\pi]$ for each $\vec{k}$ and t. The initial condition was low amplitude noise in all harmonics. Time steps were $\Delta t_{1024} = 6.7 \times 10^{-4}$ and $\Delta t_{256} = 5.0 \times 10^{-3}$. We used Fourier series in the following form:

$$\hat{F}[\eta_{\vec{r}}] = \frac{1}{(2\pi)^2} \iint_0^{2\pi} \eta_{\vec{r}} e^{i\vec{k}\vec{r}} d^2r, \quad \hat{F}^{-1}[\eta_{\vec{k}}] = \sum_{\vec{k}} \eta_{\vec{k}} e^{-i\vec{k}\vec{r}}.$$

As results of the simulation, we observed the formation of both direct and inverse cascades (Fig. 1, solid line), although exponents of powerlike spectra were different from weak turbulent solutions (4). It is important to note that development of the inverse cascade spectrum was arrested by the discreteness of the wave number grid in agreement with Refs. [18,29–31]. After that, a large scale condensate started to form. As one can see, the value of wave action $|a_k|^2$ at the condensate region is more than an order of magnitude larger than for the closest harmonic of the inverse cascade. The dynamics of large scales became extremely slow after this point. We managed to achieve a downshift of the condensate peak by one step of the wave number grid during a long time simulation on a small grid

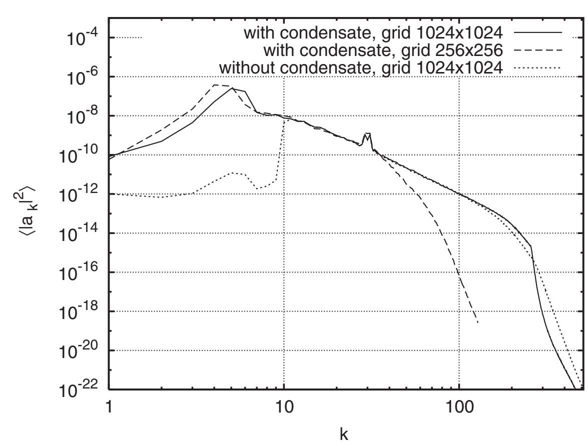

FIG. 1. Spectra $\langle |a_k|^2 \rangle$. With a condensate on the 1024×1024 grid (solid line); on the 256×256 grid with a more developed condensate (dashed line); without a condensate on the 1024×1024 grid (dotted line).

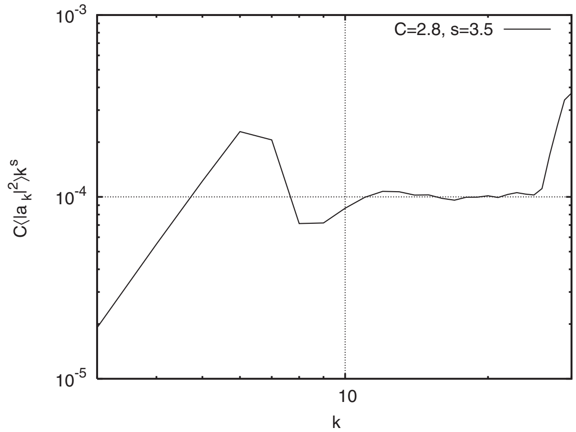

FIG. 2. Compensated inverse cascade spectra $C\langle|a_k|^2\rangle k^s$.

(256×256) (Fig. 1, dashed line). As one can see, we observed elongation of the inverse cascade interval without a significant change of the slope. Unfortunately, the inertial interval for the inverse cascade is too short to exclude the possible influence of pumping and a condensate. We can try to estimate the exponent by looking at the compensated spectra on a log-log scale (Fig. 2). The observed spectrum $\sim k^{-3.5}$ is close to a weak turbulence solution (4). The slightly lower exponent could be explained by weakening of resonant nonlinear interactions on the coarse wave number grid, which effectively decreases the homogeneity coefficient β in expression (3). For direct cascade spectra, we also used a log-log scale. Results are present in Fig. 3 (left). Formally, in this case we have quite a long inertial interval $32<k<256$, but in reality damping has an influence on the spectrum approximately up to $k\simeq180$. Still, in this case we have more than half of a decade. The theory of weak turbulence gives us dependence $\sim k^{-4}$ (3), known as the Kolmogorov-Zakharov spectrum. Nevertheless, one can see that we observe $k^{-9/2}$, known as the Phillips [1,3] spectrum. So we need to understand, what is the reason for this different spectrum slope? What changes weak turbulent theory in this case?

To answer these questions, let us compare our situation with previous works on decaying [16,19,20] or isotropic [21–23] turbulence. Immediately, we have an answer: the condensate and the inverse cascade spectrum. The inverse cascade's part of the spectrum is described by the theory of weak turbulence, so let us concentrate on the strong long ($k\simeq5$) wave influence on much shorter waves ($32<k<180$), corresponding to the direct cascade. We suppressed the condensate by including "artificial" dissipation on large scales ($k<10$). The resulting spectrum is given in Fig. 1 (dotted line). The compensated spectrum for direct cascade is given in Fig. 3 (right). As one can see, the exponent of the spectrum changed and is now closer to the results of weak turbulent theory. The small difference may be a result of the influence of the left edge of the inverse cascade, which can play the role of the condensate for short scales corresponding to the direct cascade.

A qualitative explanation of the condensate's influence on the short waves could be the following: Let us consider a propagating wave with some given slope at its front; a much longer wave can be treated as the presence of a strong background flow. If the direction of the flow is opposite to the direction of the wave's propagation, the slope of the wave's front will increase. This is what we see in our simulations. The average steepness $\mu=\sqrt{\langle|\vec{\nabla}\eta|^2\rangle}$ has changed: With the condensate $\mu_c\simeq0.14$, and without the condensate $\mu_{nc}\simeq0.12$. A more detailed picture is given by probability distribution functions (PDFs) for surface slopes (Fig. 4; also see [32]). Although the middles of the distributions are well described as Gaussians (which is one of the assumptions of the weak turbulence theory), we have significant non-Gaussian tails, and, what is more important, the widths of the PDFs are different. This means that, in the presence of the condensate, steeper waves are more probable. In nature, this will result in stronger "whitecapping": formation of a white foam cap on the crest of the wave causing additional transport of energy to the small dissipative scale. In the framework of our model, such micro-wavebreaking is impossible. Dissipation in the system prevents the formation of strong spectrum tails corresponding to the formation of discontinuities on the surface. Nevertheless, the mechanism is quite similar: Higher steepness means stronger nonlinearity in our system. In this case, for harmonics close enough to the dissipation region, the generation of second and third harmonics acts as a fast and effective additional process of energy transport to the dissipative scales. Processes corresponding to multiple harmonics generation are nonresonant, and they are neglected in the theory of wave turbulence. Also, it explains why in the experiment in the

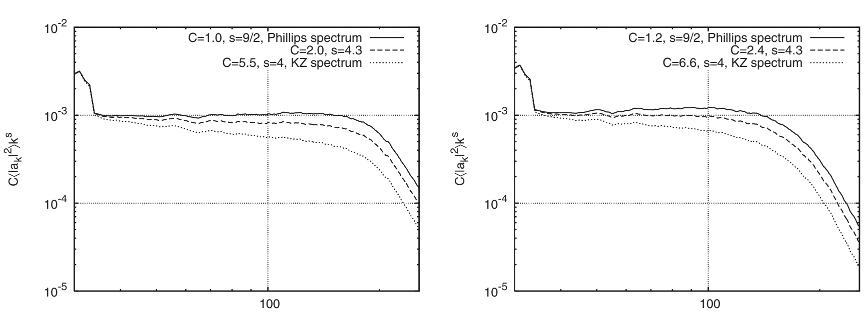

FIG. 3. Compensated direct cascade spectra $C\langle|a_k|^2\rangle k^s$ with (left) and without (right) a condensate.

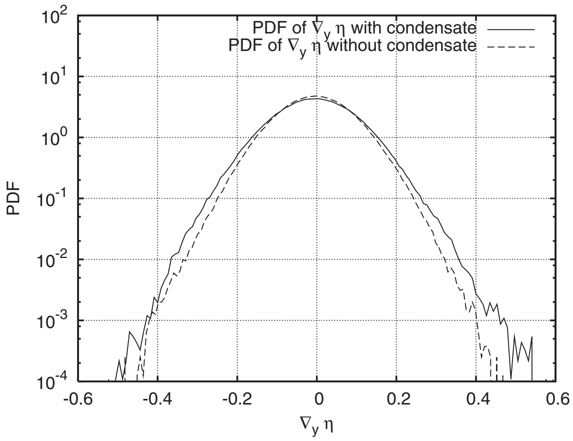

FIG. 4. PDFs of $\vec{\nabla}_y \eta$ with (solid line) and without (dashed line) a condensate.

framework of Zakharov's equations [24] spectra were nearly those of weak turbulence. Zakharov's equations take into account only resonant interactions and do not describe multiple harmonic generation. We can see that catastrophic events, such as the formation of sharp crests, which cannot be described in the statistical framework of the kinetic equation, can significantly affect the physics in the system. The wave kinetic equation can be augmented by an additional dissipation term to simulate this dissipation. As was shown in recent open field [33] and numerical [19] experiments, whitecapping dissipation is a phenomenon similar to a second-order phase transition, so even such a moderate change of the average steepness as we observed can cause significant altering of the energy transfer mechanism. Our results in the presence of a condensate support a conjecture [3] that the Phillips spectrum corresponds to a physical situation when a balance between nonlinear transport terms and intermittent dissipation takes place.

Conclusion.—In this Letter, the author presented results of the first direct numerical simulation of a direct cascade in the presence of an inverse cascade and a condensate. The importance of the condensate as a factor which increases average steepness and stimulates additional intermittent dissipation is demonstrated. A qualitative explanation of the observed spectra is given. The quantitative explanation is a subject of further investigations, because as the first step we need to create a comprehensive theory of whitecapping, which includes analysis of the fully nonlinear equations. One can use the results presented to explain observed differences in spectra in open sea and in water tank experiments.

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